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Research Article

Linear Programming process & its approach for Management Solutions

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Abstract

Linear programming (LP) is a branch or subset of mathematical programming. LP is the quantitative analysis technique for deciding to achieve the desired goal. It is a simple optimization technique. It is considered to be the most important method of optimization in different fields. It is used to obtain the most optimal solution to the problem within some constraints. It comprises of an objective function; linear inequalities subject to some constraints may be in the form of linear equations or in the form of inequalities. This method is used to maximize or minimize the objective function of the given mathematical model comprising the set of linear inequalities depending upon some constraints represented in the linear relationship. The aim of this research is to try to address a mathematical methodology used in the field of assisting in taking management decisions of a quantitative multi-objective nature, mainly represented by the linear programming model.

Keywords: *Linear Programming, Mathematical function, Klee – Minty, Decision variable, Optimization*

1. Introduction

Linear programming is the name of a branch of applied mathematics that deals with solving optimization problems of a particular form. Linear programming problems consist of a linear cost function (consisting of a certain number of variables) which is to be minimized or maximized subject to a certain number of constraints. The constraints are linear inequalities of the variables used in the cost function.

The cost function is also sometimes called the objective function. Linear programming is closely related to linear algebra; the most noticeable difference is that linear programming often uses inequalities in the problem statement rather than equalities. LP is concerned with the maximization or minimization of a linear objective function in many variables subject to linear equality and inequality constraints. It deals with the optimization of linear functions subject to linear constraints. Particularly, LP helps to allocate the resources efficiently to profit maximization, loss minimization, or to utilize the production capacity to the maximum extent. LP always helps formulate the real-life problems into a fixed mathematical model. Thus, it is also defined as a mathematical technique for solving different real-life problems to determine the optimal solution or for calculating the best value within the context or situation. It is used to find the optimum utilization of the resources at the minimum cost. In another words, it deals with the allocation of resources with some restrictions such as costs and availability.

Operations research models are among the most important models that help in making decisions as they depend on the scientific method in solving problems and deal with different aspects of scientific management.

Mathematical programming is defined as the science that seeks to determine the maximum or minimum value of a specific function called the objective function, which depends on a finite number of variables. These variables may be independent of each other, or they may be related to each other by what is called constraints.

1. Assuming that the objective function and all the constraint functions are linear functions, then this system is called linear programming.
2. Linear programming is called Integer Programming if the variables are integers.
3. Assuming that the objective function or the constraints or both are non-linear functions, then this system is called non-linear programming.

2.Operations Research

Operations research is classified as a branch of applied mathematics, and it is known by several names, including decision science or mathematical

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programming, and this branch focuses on the processes and methods of finding optimal solutions to the problems facing the decision-maker in establishments and working to improve them, and it intervenes in several areas such as engineering applications and economic sciences Among the most prominent methods used in operations research are the methods of mathematical modeling and statistical analysis, which are often used for the purposes of reaching the optimal decision-making among a set of alternatives presented.

It is worth noting that operations research has special systems that you depend on in determining the best available results and making decisions based on them, and among these systems are: linear programming or known as Simplex, nonlinear programming, dynamic programming, target programming, integer programming, and probabilistic programming, where the choice is made. The most appropriate system according to the case studied, and we will focus the light in this article on linear programming in order to know what it is, its areas of application, and the terms and steps of its use. Because in fact the simplified method exists for solving real world problem and the behavior of the simplified algorithm is the behavior of polynomial time algorithm in solving such problems, and we rarely meet in reality Klee - Minty problems.

3. Linear programming

It is a basic and important method that helps decision-makers to make correct decisions in a scientific way. Linear programming problems are part of mathematical programming problems that include both linear and nonlinear. Moreover, mathematical programming is, in turn, part of a more comprehensive subject, called operations research or operational research, which all concern matters of organization, management, transportation, agriculture, industry, etc. Linear mathematical programming is a matter of preference, and here by preference problems we mean those mathematical problems that seek to maximize or reduce a linear (dependent) function placed into linear mathematical constraints as well.

4. History of linear programming

Linear programming appeared in 1947 CE, especially after World War II, by the mathematician George B. Dantzig, who was working as an expert in the US Army, and in 1949 CE, George Danzig published the Simplex Algorithm for solving linear programs (linear problems), and since this time, contributions have poured in Improving linear software solution in new ways. Although the Simplex Method for solving linear programs is considered an exponential time algorithm by Klee-Minty, yet this method is considered to be one of the top 10 algorithms in mathematics at all, why? Because in fact the simplified method exists for solving real world problem and the behavior of the simplified algorithm is the behavior to polynomial time algorithm in solving such problems, and we rarely meet in reality Klee - Minty problems.

Although the Simplex Method for solving linear programs is considered an exponential time algorithm by Klee-Minty, yet this method is considered to be one of the top 10 algorithms in mathematics at all.

5. Formula & concept for Linear Programming

We noticed from the previous models that the purpose of linear programming is to find the minimum or maximum value of a linear function (called the objective function) whose variables are subject to linear conditions in the form of inequalities or equations. Whatever the wording of the linear program, it can be expressed in all cases in the following standard form

$$\min c^1x^1 + c^2x^2 + \cdots + c_n x_n$$

s.t

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$

$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

whereas $x_i \geq \ , i = 1,2,3,\dots,_n$

And we mean that the linear system above is a system of linear equations in standard form, and the coefficients aij, bj, ci are fixed numbers, while, x_i are the variables to be assigned. The standard formulation of the linear system above can be briefly expressed as *follows:*

$$\min \quad f(x) = c^T x$$
s.t
$$Ax = b \ : x \geq 0$$

Note: If the goal function is maximization, it can be returned to a minimization function by making z' = -z.

Let us have the following linear program:

max $z = 40c\ 50t$
s.t

$$2c + 4t \leq 60$$
$$3c + t \leq 75$$
$$c + 4t \leq 84$$

where the limits c ≥ 0, t ≥ 0

Which can be transferred to the following linear program:

min z' = -40c + 4t ≤ 60
s.t

$$2c + 4t \le 60$$
$$3c + t \le 75$$
$$c + 4t \le 84$$

Where the limits c ≥ 0, t ≥ 0.
The greatest value of z corresponds to the smallest value of, so the greatest value of z equals negatives the smallest value of. The two issues have the same optimal solution.

The slack variables

If the linear program is formatted as follows:

$\min c_1x_1 + c_2x_2 + \dots. + c_nx_n$
$s.t$
$$a_{11}x_1 + a_{12}x_2 + \dots. + a_{1n}x_n \le b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots. + a_{2n}x_n \le b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots + a_{mn}x_n \le b_m$$

Where, i = 1, 2, 3, ..., n, xi ≥ 0

It can be returned to the standard format, by introducing new variables

$$xn + 1, xn + 2, \dots., xn + m$$

We call them complementary variables, so the inequalities are transformed into equations, which make the number of unknowns greater than the number of equations, and therefore there is no single solution to the previous set of equations, and the new form of the linear program is as follows:

$$min\ c_1x_1 + c_2x_2 + \cdots\dots + c_nx_n$$
$$s.t.$$
$$a_{11}x_1 + a_{12}x_2 + \dots. + a_{1n}x_n + xn + 1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots. + a_{2n}x_n + xn + 2 = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots. + a_{mn}x_n + xn + m = b_m$$

$$where, i = 1, 2, 3..., n + m, x_i \ge 0$$

The surplus variables

It is an opposite case to the previous case, and the linear program is formulated as follows:

$$min\ c_1x_1 + c_2x_2 + \dots. + c_nx_n$$
$$s.t.$$
$$a_{11}x_1 + a_{21}x_2 + \dots. + a_{1n}x_n \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots. + a_{1n}x_n \ge b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots. + a_{mn}x_n \ge b_m$$

$$where, i1, 2, 3 \dots., n, \quad x_i \ge 0$$

It can be returned to the standard format, by introducing new variables

$$xn + 1, xn + 2, \dots., xn + m$$

We call them extra variables and inequalities turn into equations, which makes the number of unknowns greater than the number of equations, and therefore there is no single solution to the previous set of equations, and the new form of the linear program is as follows:

$$min\ c_1x_1 + c_2x_2 + \dots. + c_nx_n$$
$$s.t.$$
$$a_{11}x_1 + a_{12}x_2 + \dots. + a_{1n}x_n - xn + 1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots. + a_{2n}x_n - xn + 2 = b_2$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots. + a_{mn}x_n - xn + m = b_m$$

$$where\ I = 1, 2, 3, \dots., m + n, \quad x_i \ge 0$$

Free variables

If the linear program was written in the standard form but one of the xj variables was not supposed to be non-negative, that is, it is a free variable, then there is a way to convert the linear program to the standard form. We review the following:

The variable xj can be replaced with two non-negative variables uj, vj by writing

$$xj = uj - vj$$

Thus, the linear program is now represented by the n + 1 variables, which are:

$x_1, x_2, x_3, \dots., xj, xj + 1, \dots, x_n$
After deleting xj and adding the two non-negative uj, vj variables to the linear program.

6.Simplified algorithm (tabular form method)

1-We begin by formulating the linear program problem in standard form.
2-We put the information arising from the problem in a table, make the last line of the table represent the objective function and denote the coefficients in the last row with rj and call them the relative cost coefficients.
3-We repeat the following:
• We designate the entering variable to the base by examining the numbers in the last row that represent r. If all (except for the right number) are positive or zeros, then the table is considered final and we get the optimal solution from it, and if not, then we take the smallest of these numbers Take into account and call its column the pivot column (if the objective function in the linear program is a reduction function.
• We define the leaving variable from the base according to the formula called ratio test and based on the smallest ratios arising from dividing the numbers of the right column by the positive numbers corresponding to it in the specified column that was set in the previous step. If we do not find a positive

number in the pivot column that means that the problem is infinite

4- We perform pivotal operations

• Divide each number of the pivot row numbers by the pivot element (the pivot element is the intersection of the pivot row and the pivot column).

• Subtract appropriate multiples of the new pivot row from the numbers of the other rows, so that each element of the pivot column (except for the pivot element) becomes equal to zero.

Linear program by simplex method

$$min\ z = -3x_1 + 3x_2 + 5x_3$$
$$s.t.$$
$$2x_1 - 2x_2 + 3x_3 \leq 1$$
$$x_1 - x_2 - 2x_3 \leq 1$$
$$x_1, x_2, x_3 \geq 0$$

First: We begin by formulating the linear program of the problem into the standard form by entering the following complementary variables as follows:

$$\min \quad z = -3x_1 + 3x_2 + 5x_3$$
$$s.t.$$
$$2x_1 - 2x_2 + 3x_3 + x_4 = 1$$
$$x_1 - x_2 - 2x_3 + x_5 = 1$$
$$x_1, x_2, x_3, x_4, x_5 \geq 0$$

Second: We put the information arising from the issue in the following table:

x_4	2	2-	3	1	0	1
x_5	1	1-	2-	0	1	1
z	3-	3	5	0	0	0

Third and fourth: By searching the objective function, we find that the smallest number is the number -3 corresponding to the variable x1, and then the variable x1 is the variable entering the base entering variable. / 2 corresponding to the variable x4, and then we find that the leaving variable is $x4$, so the pivot element is the number 2.

Fourth: We perform pivotal operations so that the pivot element is equal to a true one, and the elements that lie below it and that lie above it (if any) are equal to zero as follows.

x_1	1	1-	1.5	0.5	0	0.5
x_5	0	0	3.5-	0.5-	0	0.5
z	0	0	9.5	1.5	0	1.5

Then by looking again in the elements of the objective function, you find that they are all non-negative, and then we find that the objective function becomes $z = 1.5\ as\ x1 = 0.5, x5 = 0.5$

Notes on the com of Sim

The analysis of the implementation time of the simplified algorithm is very complex due to the scarcity of results available. No one knows "or in other words, no scientist in linear programming came out to us and tells us the best pivotal strategy for choosing the pivotal element, " and because there are no results that tell us the number of steps expected for a particular' class of linear programming problems.

It is possible to create artificial examples (rare in our daily life) of the implementation time of the simplified algorithm as an exponential function of the number of variables, and among these examples are Klee - Minty examples which we will show later.

Although the complexity of the simplified algorithm is an exponential function in the number of variables, however, this algorithm is considered one of the 10 best algorithms in mathematics and computer because its behavior behaves like polynomial algorithms in the number of variables to solve life problems. As for the fact that Klee - Minty examples are rare in reality.

Note: Those who deal with scientific research using the Simplex Method to obtain new and really good results must classify linear programming problems into categories and conduct their research methods (different pivotal strategies to define the pivotal element) and record their results.

Klee - Minty matters

In 1973 Victor Klee and George J. Minty proved that the simplified algorithm is not a polynomial algorithm in time but an exponential algorithm, and here is the following rule that explains this.

$$max \quad \sum_{j=1}^{n} 2^{\mathrm{n-j}}\, xj$$
$$s.t.\ \ x_1 \leq 5$$
$$\sum_{j=1}^{i-1} 2^{\ \mathrm{i-j}}\ xj + x_{\mathrm{i}} \quad \leq 5^{\mathrm{i}}$$
$$i = 2 \ldots\ldots, n,$$
$$xj \geq 0, \qquad j = 1, \ldots., n$$

A Klee-Minty Example

Klee-Minty example number n is

maximize $\sum_{j=1}^{n} 10^{\mathrm{n-j}} xj$

subject to $2\sum_{j=1}^{\mathrm{i-1}} 10^{\mathrm{i-1}}\ xj + x_{\mathrm{i}} \leq 100^{\mathrm{i-1}}\ for\ 1 \leq i \leq n$ all $xj \geq 0$

Using the most-negative-entry pivoting rule, this require $2^{\mathrm{n}} - 1$ p:

find an optimal solution.

In the case n = 3, the prol leni is:

maximize $100x_1 + 10x_2 + x_3$

subject to $x_1 \leq 1$

$$20x_1 + x_2 \leq 100$$
$$200x_1 + 20x_2 + x_3 \leq 10000$$
$$x_1, x_2, x_3 \geq 0$$

Initial tableau:

Z	X1	X2	X3	S1	S2	S3	rsh		
1	-100	-10	-1	0	0	0	0	=	z

0	1	0	0	1	0	0	1	=	s_1
0	20	1	0	0	1	0	100	=	s_2
0	200	20	1	0	0	1	10000	=	s_3

First pivot: x_1 enters, s_1 leaves the basis.

Z	X1	X2	X3	S1	S2	S3	rsh		
1	0	-10	-1	100	0	0	100	=	z
0	1	0	0	1	0	0	1	=	x_1
0	0	1	0	-20	1	0	80	=	s_2
0	0	20	1	-200	0	1	9800	=	s_3

Second pivot: x_2 enters, s_2 leaves

Z	X1	X2	X3	S1	S2	S3	rsh		
1	0	0	-1	-100	10	0	900	=	z
0	1	0	0	1	0	0	1	=	x_1
0	0	1	0	-20	1	0	80	=	s_2
0	0	0	1	200	-20	1	8200	=	s_3

Third pivot: s_1 enters, x_1 leaves

Z	X1	X2	X3	S1	S2	S3	rsh		
1	100	0	-1	0	10	0	1000	=	z
0	1	0	0	1	0	0	1	=	s_1
0	20	1	0	0	1	0	100	=	x_2
0	-200	0	1	0	-20	1	8000	=	s_3

Fourth pivot: x_3 enters, s_3 leaves

Z	X1	X2	X3	S1	S2	S3	rsh		
1	-100	0	0	0	-10	1	9000	=	z
0	1	0	0	1	0	0	1	=	s_1
0	20	1	0	0	1	0	100	=	x_2
0	-200	0	1	0	-20	1	8000	=	s_3

Fifth pivot: x_1 enters, s_1 leaves

Z	X1	X2	X3	S1	S2	S3	rsh		
1	0	0	0	100	-10	1	9100	=	z
0	1	0	0	1	0	0	1	=	s_1
0	0	1	0	-20	1	0	100	=	x_2
0	0	0	1	200	-20	1	8200	=	s_3

Sixth pivot: s_2 enters, x_2 leaves

Z	X1	X2	X3	S1	S2	S3	rsh	Z	X1
1	0	10	0	-100	0	1	9900	=	z
0	1	0	0	1	0	0	1	=	x_1
0	0	1	0	-20	1	0	100	=	s_2
0	0	20	1	-200	0	1	9800	=	x_3

Seventh pivot: s_1 enters, x_1 leaves

Z	X1	X2	X3	S1	S2	S3	rsh	Z	X1
1	100	10	0	0	0	1	10000	=	z
0	1	0	0	1	0	0	1	=	s_1
0	20	1	0	0	1	0	100	=	s_2
0	200	20	1	0	0	1	10000	=	x_3

This is optimal

7.Special cases for linear programming issues

1. Non-uniformity of multiple optimal solutions

If all numbers in the last row are non-negative and there is a zero value corresponding to a non-fundamental variable, then this means that there is more than one optimal solution, meaning that the solution is not unique.

2. Unbounded objective function

If all numbers in the pivot column are not positive, the objective function is infinite.

3. Degenerate Solution case

Usually, this condition results when we get ratios in the Ratio Test equal to zero. Often in this case, we have abundance and an increase in restrictions, and perhaps to overcome this problem we can remove a set of excessive restrictions.

4. The state of Cycling

In very rare cases, the condition of the irregular solution may lead to Cycling.

This occurs when there is more than one variable leaving the base. It was overcome by Bland in 1977 that its simple rule prevents the phenomenon of spinning. This rule is summarized as follows: Assuming that we have more than one variable leaving the basis, the selection is made on the basis of the least evidence of variables

Conclusion

Linear programming is an important branch of applied mathematics that solves a wide variety of optimization problems. It is widely used in production planning and scheduling problems. Many recent advances in the field have come from the airline industry where aircraft and crew scheduling have been great improved by the use of linear programming. The revised simplex method is not theoretically satisfactory from a computational point of view, it is by far the most widely used method to solve linear programming problems and only rarely are its limitations encountered in practical applications. Simplex Method for solving linear programs is considered an exponential time algorithm by Klee-Minty, yet this method is considered to be one of the top 10 algorithms in mathematics. The biggest advantage of linear programming as an optimization method is that it always achieves the optimal solution if one exists.

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