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Research Article

Isoparametric Elements applied to Curved Boundaries

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Abstract

The word isoparametric is used when the same shape functions called as interpolation functions [N] to define the element's geometric shape as well as used to define the displacements within the element. The Isoparametric element equations are formed by using an intrinsic coordinate systems i.e. natural coordinate system that is defined by element geometry and It is not defined by the orientation of the element in the global-coordinate system. In this paper the Isoparametric formulation for simple bar element is explained first and it will lead to a simple mathematical computation. Then the Numerical Integration for Gauss Quadrature for two and three sampling points is explained.

Keywords: *Isoparametric, Interpolating functions, Gauss Quadrature, Numerical Integration, etc.*

1. Introduction

Mostly 2D and 3D elements having regular geometry such as triangular and rectangular element are having straight edges. Therefore for the analysis of any irregular geometry, it is difficult to use such elements directly. For example, if we see the continuum having curve boundary as shown in the Fig. 1(a) It has been discretized into a mesh of finite elements in three different ways given by figure 1.

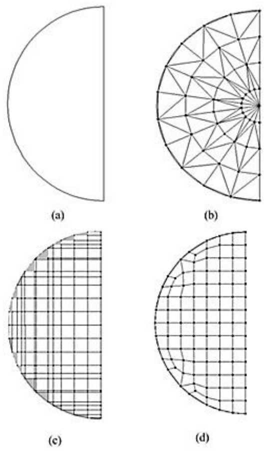

Fig.1(a)The Continuum to be discritized (b),(C),(d) Discritization using Triangular, rectangular and combination of rectangular and quadrilateral elements

Figure 1(b) gives a possible mesh by using triangular elements. As we know that the triangular elements can suitable approximate the circular boundary of the continuum, but the elements very close to the center area will become the slender and thus affect the accuracy of finite element problem solution. We can also reduce the height of each row of elements as we approaching towards the center zone. But, unnecessary refining of the continuum body creates relatively very large number of elements and will lead to increase in the computation time. Alternatively, when we do the meshing using rectangular elements as given in Fig 1(c), the area of continuum body excluded from the finite element model. It is significantly sufficient to give incorrect results. To improve the accuracy of the result we can generate mesh by using very small elements. But, this will again significantly increases the computation time. Another way by using both rectangular and triangular elements. But such types of combination will not provide the best solution in terms of accuracy, as the different order polynomials are used to give the field variables for different types of elements. Also the triangular elements may be slender and therefore affect the accuracy. The same continuum body is discritized with rectangular elements near center as shown in figure 1(d) with four-node quadrilateral elements near boundary. By using the concept of mapping this four-noded quadrilateral element can be derived from rectangular elements. Using the concept of mapping regular triangular, rectangular or solid elements in natural coordinate system can be transformed into global Cartesian coordinate system having arbitrary shapes with curved edge or surfaces.

The natural coordinate system is called as parent element system. Fig. 2 gives the parent elements in intrinsic i.e. natural coordinate system and the mapped

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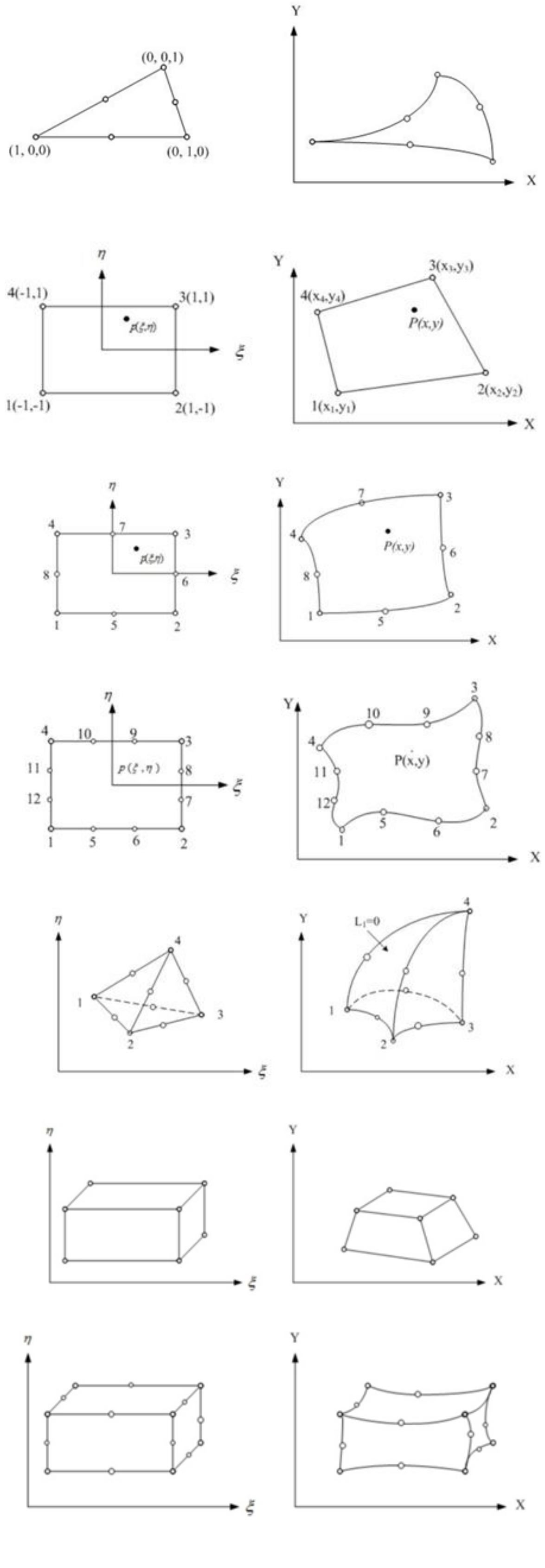
elements used in the global Cartesian co-ordinate system (S. S. Bhavikatti,2015).



a)Natural b) Global

Fig.2 Parent Elements in Natural co-ordinates and mapping in Global Co-ordinate system

2. Meshing

This part gives a method or a way to generate high-order meshes cosisting of highly stretched elements. The literature gives the generation of a high-order coarse mesh having prisms at the boundary and having tetrahedra . After that we can describe the creation of finer mesh by splitting the coarse prismatic elements hereafter mentioned as 'macro-elements' into finer elements, which utilizes the mapping that will defines each macro-element to insert the necessary curvature of a boundary into sub-elements. Afterthat we describe the way this technique be further adapted to gives meshes that are having only tetrahedral elements. The various steps involved are described in the following sections (D. Moxey, M.D. Green, *et al,* 2014).

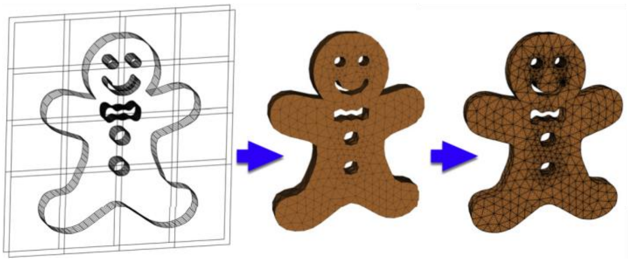


Fig.3 High order Meshing Strategy.

3. Isoparametric Formulation of the Bar Element

The isoparametric method is seen to be some what tedious and quite confusing initially, but it will lead to a simple computer program formulation. This methiod is generally applicable for two-and three-dimensional stress analysis and for nonstructural problems. And it will gives the good results for 2D and 3D structural problems. The isoparametric element formulation allows elements to be created that are non rectangular and may have curved sided boundaries. There are number of commercial computer programs as described earlier have adapted this formulation for the various libraries of elements (Daryil Logan 2007).

Use of the bar element makes gives relatively easy and simple approach for the understanding of the Isoparametric Element Formulation technique. Because as it will lead the simple expressions result. Then, we will consider the development of the isoparametric formulation of the simple quadrilateral element stiffness matrix. Also.

The term isoparametric has been derived from the concept that using same shape functions (or interpolation functions) [N] to define the geometry of the element as well as are used to define the displacements within the element. Thus, when the interpolation function is u = a1 + a2s for the displacement, we use x = a1 + a2s for the description of the nodal coordinate of a point on the bar element and, hence, the physical shape of the element (S. S. Bhavikatti 2014).

Isoparametric element equations are formulated by using a natural or intrinsic coordinate system 's' which is defined by element geometry and not by orientation of the element in the global cartesian co-ordinate system. In other words, axial coordinate 's' is attached to the bar and remains directed along the axial length of the bar, regardless of how the bar is oriented in space. There is a relationship known as transformation mapping between the natural coordinate systems and the global catesian coordinate system x for each and every element of a specific structure.

Firstly the natural coordinate 's' is attached to the element with the origin located at the center of the element. The 's' axis need not to be parallel to the x axis this is only and only for the convenience. Consider the bar element to have two degrees of freedom-axial displacements u1 and u2 at each node associated with the global x axis (Daryil Logan 2007).

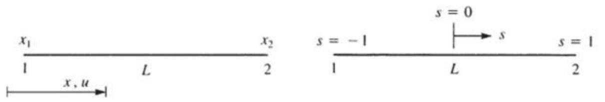


Fig.4 Transformation Mapping for 1-D Element

For the special case when the two axes 's' and 'x'are parallel to each other, the s and x-coordinates can be related and given by equation 1.

$$x = x_c + \left(\frac{L}{2}\right) s \quad \text{(Eqn.1)}$$

where x_c is the global coordinate of the element centroid. Using the global coordinates x_1 and x_2 in above equation with $x_c = (x_1 + x_2)/2$, we can express the natural coordinate s in terms of the global coordinates as

$$s = \left[x - \frac{x_1+x_2}{2}\right][2/(x_2 - x_1] \quad \text{(Eqn.2)}$$

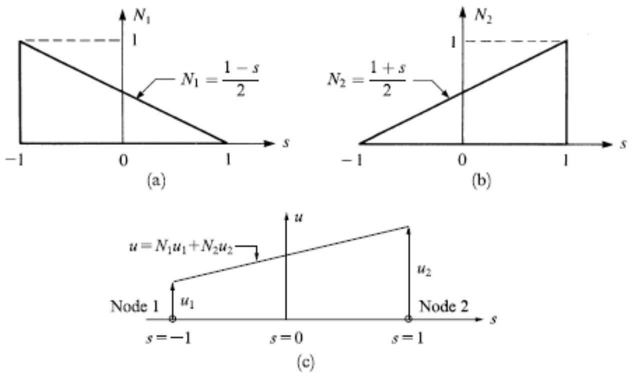


Fig.5 Shape function variations with natural coordinates: (a) N1, (b) N2, and (c) linear displacement field u plotted over element length (Daryil Logan 2007).

The shape functions which is used to define a position within the bar element are found in a similar manner to that of used previously to define displacement within a bar. Then We begin to relate the natural coordinate system to the global cartrsian coordinate system and given by,

$$x = a_1 + a_2 s$$

where it is to be noted that s is such that $-1 \le s \le 1$. Solving for the ai's in terms of x1 and x2, we obtain

$$x = 1/2[(1-s)x_1 + (1+s)x_2] \quad \text{(Eqn.3)}$$

Or in Matrix form

$$\{x\} = [N1 \quad N2]\begin{Bmatrix} x1 \\ x2 \end{Bmatrix} \quad \text{(Eqn.4)}$$

Shape functions used are

$$N_1 = \frac{1-s}{2} \qquad N_2 = \frac{1+s}{2} \quad \text{(Eqn. 5)}$$

The linear shape functions in above Eqs. map the 's' coordinate of any point in the element to the x-coordinate when used in Eq. (3).

For instance, when we substitute s =-1 into Eq. (2), we obtain x= x_1. These shape functions are shown in Figure(5), where we may see that they have the same properties as defined for the interpolation functions. Hence, N1 represents the physical shape of the coordinate x when plotted over the length of the element for x_1=1 and x2=0, and N2 represents the coordinate x when plotted over the length of the element for x_2=1 and x_1=0. Again, we must have N1+N2=1.

These shape functions should be continuous through out the element domain. Also they must have finite first derivatives within the element.

The displacement function {u} within the bar is derived by using the same shape functions. Eqs(5) as are used to define the element shape; that is,

$$\{u\} = [N1 \quad N2]\begin{Bmatrix} u1 \\ u2 \end{Bmatrix} \quad \text{(Eqn.6)}$$

When a particular coordinate s of the point of interest is substituted into [N], Eq.(6) yields the displacement of a point on the bar in terms of the nodal degrees of freedom u1 and u2 as shown in Figure 5(c). As u and x are defined by the same shape functions at the same nodes, comparing Eqs. (4) and (6), the element is called isoparametric.

4. Numerical Integration

Gaussian Quadrature Formula and Newton's Quotes Formula This point describe Gaussian's method, one of the many methods used for numerical calculation of definite integrals. As it has proved to be the most useful for finite element work. For completion sake, we will also describe the more common numerical integration method of Newton-Cotes. The Newton-Cotes methods for one and two intervals of integration are the well-known trapezoid and Simpson's one-third rule, respectively.

After describing both methods, we can easily understand why the Gaussian quadrature method is most useful in finite element work (Daryil Logan 2007).

Gaussian Quadrature Method

To evaluate the integral

$$\int_{-1}^{+1} y\,dx \qquad \text{Eqn. (7)}$$

where y= y(x), we might choose a sample or we may evaluate the y at the midpoint y(0)= y1 and multiply by the length of the interval, as shown in the Figure 6 to arrive at I=2y1. The exact result the curve turns to straight line. And we can say that this is an example of what is called one-point Gaussian quadrature because only one sampling point was used. Therefore,

$$I = \int_{-1}^{+1} y(x)dx \cong 2y(0) \qquad \text{Eqn. (8)}$$

which is nothing but the familiar midpoint rule.

Generalization of the formula may leads to

$$I = \int_{-1}^{+1} y(x)dx = \sum Wiyi \qquad \text{Eqn. (9)}$$

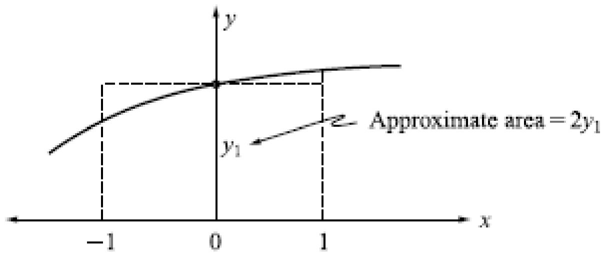


Fig.6 Gaussian quadrature using one sampling point

That is, to approximate the integral, we evaluate the function at several sampling points n, multiply each value yi by the appropriate weight Wi, and add the terms. Gaussians method chooses the sampling points such that for a given number of points, the best possible accuracy is obtained. Sampling points are located symmetrically with respect to the center of the interval. Symmetrically paired points are given the same weight W_i.

Following Table gives appropriate sampling points and weighting coefficients for the first three order that is, one, two, or three sampling points. Table for Gauss points for integration from minus one to one $\int_{-1}^{+1} y(x)dx = \sum Wiyi$ Where summation is from i=1 to i=n.

Number of Points	Locations, x_i	Associated Weights, W_i
1	$x_1 = 0.000\ldots$	2.000
2	$x_1, x_2 = \pm 0.57735026918962$	1.000
3	$x_1, x_3 = \pm 0.77459666924148$	$\frac{5}{9} = 0.555\ldots$
	$x_2 = 0.000\ldots$	$\frac{8}{9} = 0.888\ldots$
4	$x_1, x_4 = \pm 0.8611363116$	0.3478548451
	$x_2, x_3 = \pm 0.3399810436$	0.6521451549

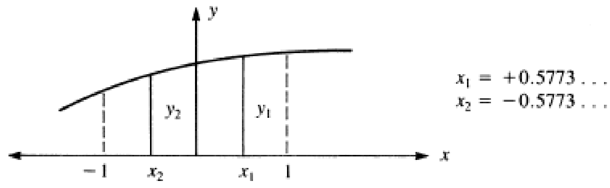


Fig.7 Gaussian quadrature using two sampling point

Two-Point Formula

To express the derivation of a two-point (n=2) Gauss formula based on Eq. 9 we have,

$$I = \int_{-1}^{+1} ydx = W_1y_1 + W_2y_2 = W_1y(x_1) + w_2y(x_2)$$

Conclusion

Isoparametric formulation is very important for the meshing of curved boundaries. We can use this technique for various modeling applications such as aero plane bodies, wind turbine blades, rotors etc. Also the Gaussian Quadrature 2-point and 3-point formulae can be effectively used to find out the numerical integration.

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