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Research Article

Forced Monotone Methods

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Abstract

We consider a two point conjugate boundary value problem of the form $y''(t) = f(t, y(t), y'(t))$, $a \le t \le b$, $y(a) = a_1$, $y(b) = a_2$, where $a < b$, $f: [a,b] \times R^2 \longrightarrow R$ is continuous and a_1 and a_2 are real. The method of upper and lower solutions, coupled with monotone methods, is useful if f is independent of y'. If the conjugate conditions, $y(a) = a_1, y(b) = a_2$, are replaced by right focal conditions $y(a) = a_1, y'(b) = a_2$, then the method of upper and lower solutions, coupled with monotone methods, is useful in the case that f depends on y and on y'. In this talk, we construct a boundary value problem of the form $y''(t) = f(t, y(t), y'(t))$, $a \le t \le b$, $y(a) = a_1$, $y'(b) = g(y, y')$, which is equivalent to the original two point conjugate problem and obtain sufficient conditions on f and on g such that the method of upper and lower solutions, coupled with monotone methods, is useful.

Keywords: *Forced Monotone Methods etc.*

1. Introduction

We are interested to obtain sufficient conditions for the existence of solutions of second order of nonlinear differential equations by using the method of upper and lower solutions coupled with the monotone method. In this paper, we consider three nonlinear equations with boundary values problems,

$$y''(t) = f(t, y(t), y'(t)),\ 0 \le t \le 1,\ y(0) = A, y(1) = B \quad (1)$$

and

$$y''(t) = f(t, y),\ 0 \le t \le 1,\ y(0) = A, y(1) = B \quad (2)$$

and

$$y''(t) = f(t, y, y'),\ 0 \le t \le 1, y(0) = A, y'(1) = B \quad (3)$$

In (2), the Green's function is of constant sign and f depends only on y, so the method of upper and lower solution coupled with monotone method can be applied. In (3), the Green's function and its partial derivative are of constant sign and f depends on y and y′, so the method of upper and lower solution coupled with monotone methods can be applied. In (1), the partial derivative of the Green's function changes sign and f depends on y′, so the method of upper and lower solution coupled with monotone method does not readily apply. We explore method to force monotone convergence of iterates of upper and lower solutions.

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In what follows, we first show in detail how the method of upper and lower solutions, coupled with monotone methods apply to (2); we also briefly out line to method applied to (3). We propose a method of forced monotonicity and apply it to a special case of (1).

An informative description of monotone operators is found in (1); details related to Green's functions and fixed point theorems are found in (2).

2 The method of Monotone Operators

We have the boundary value problem for the second order differential equation

$$y''(t) = f(t, y(t), y'(t)),\ 0 \le t \le 1,\ y(0) = A, y(1) = B, \quad (1)$$

$f: [0,1] \times R^2 \rightarrow R$is continuous map.

- Methods to give an analytic solution to (1) do not exist. We give sufficient conditions so a solution exists.
- Methods of upper and lower solution, coupled with Nagumo conditions, give sufficient conditions for existence of solution of (1).

Nagumo Condition

$$y''(t) = 1 + (y')^2,\ y(0) = A, y(\pi) = B$$

No solution exists. For this differential equation the initial value problem with initial values $y(0) = A$, $y'(0) = m$ has unique solution

$$y = A + \ln\sec(\tan^{-1} m) - \ln\sec(\tan^{-1} m)$$

- We will motivate this talk with an application of method of upper and lower solutions, coupled with monotone methods to

$$y''(t) = f(t,y),\ 0 \le t \le 1,\ y(0) = A, y(1) = B,\ (2)$$

where $f: [a,b] \times R \to R$ is continuous. Assume in addition, f is decreasing in y for fixed t.

Note: equation (2) has no dependence on y'.

Define $K : C[0,1] \to C[0,1]$ by

$$Ky(t) = A + (B-A)t + \int_0^1 G_0(t,s) f(s,y(s))ds, \qquad 0 \le t \le 1$$

where $G_0(t,s) = \begin{cases} t(s-1), & \text{if } 0 < t < s < 1, \\ s(t-1), & \text{if } 0 < s < t < 1. \end{cases}$

Theorem2.1 y is a solution of (2) $\Leftrightarrow y \in C[0,1]$ and $y = Ky$

$G_0(t,s) \le 0$ on $[0,1] \times [0,1]$,so with f is decreasing in y, if $y_1(t) \le y_2(t),\ 0 \le t \le 1,$

then

$$\begin{aligned} Ky_1(t) &= A + (B-A)t + \int_0^1 G_o(t,s) f(s, y_1(s))ds \\ &\le A + (B-A)t \\ &+ \int_0^1 G_0(t,s) f(s,y_2(s))ds = Ky_2(t) \end{aligned}$$

Note: K is a monotone operator.

Definition 2.2 Definition of the upper and the lower solution: Assume existence of $w_0, v_0 \in C^2[0,1]$ satisfying:

- $w_0(t) \le v_0(t),\ 0 \le t \le 1,$
- $w_0(0) \le A \le v_0(0),\ w_0(1) \le B \le v_0(1),$
- $w''_0(t) \ge f(t, w_0(t)), v''_0(t) \le f(t, v_0(t)), 0 \le t \le 1.$

Define $\{w_n\}, \{v_n\}$ by

$$w_{n+1}(t) = Kw_n(t),\ \ v_{n+1}(t) = Kv_n(t), \qquad n = 0,1, \dots \dots$$

- Recall $y \in C[0,1]$ is a solution of

$$y'' = f(t, y(t)), \qquad y(0) = A, \qquad y(1) = B \Leftrightarrow y = Ky$$

Our goal is to show $w_n$converges in $C[0,1]$ to some w. Then $w = Kw$ and the limit is a solution of (2).

Consider

$$y''(t) = w''_0(t),\ y(0) = w_0(0),\ y(1) = w_1(1)$$

Then

$$\begin{aligned} w_0(t) &= w_0(0) + \big(w_0(1) - w_0(0)\big)t \\ &+ \int_0^1 G_0(t,s) w''_0(s)ds \end{aligned}$$

So,

$$\begin{aligned} w_0(t) &\le A + (B-A)t \\ &+ \int_0^1 G_0(t,s) f(s, w_0(s))ds = w_1(t) \end{aligned}$$

Similarly, $v_0(t) \ge v_1(t)$

So,

$$w_0 \le v_0 \Longrightarrow w_1 \le v_1$$

and

$$w_0 \le w_1, \qquad v_1 \le v_0$$

Thus,

$$w_0 \le w_1 \le v_1 \le v_0$$

- Since K is monotone,

$$w_n \le w_{n+1} \le v_{n+1} \le v_n,\ \ n = 0,1, \dots \dots$$

$$w_n \uparrow w \in C[0,1], \qquad v_n \downarrow v \in C[0,1],$$

and

$$w = Kw, \qquad v = Kv$$

- For further motivation consider,

$$y''(t) = f(t,y,y'),\ 0 < t < 1,\ y(0) = A,\ \ y'(1) = B, (3)$$

Where$f: [0,1] \times R^2 \to R$ is continuous and $\frac{\partial f}{\partial y} \le 0,\ \frac{\partial f}{\partial y'} \le 0$ on $[0,1] \times R^2$.

Here,

$$Ky(t) = A + Bt + \int_0^1 G_1(t,s) f(s, y(s), y'(s))ds$$

where $K: C^1[0,1] \to C^1[0,1]$, is defined by

$$G_1(t,s) = \begin{cases} -t, & \text{if } 0 < t < s < 1, \\ -s, & \text{if } 0 < s < t < 1. \end{cases}$$

$$G_1(t,s) \le 0 \text{ on } [0,1] \times [0,1]$$
$$\frac{\partial}{\partial t} G_1(t,s) \le 0 \text{ on } [0,1] \times [0,1]$$

$y_1 \le y_2$means that

$$y_1(t) \le y_2(t), \qquad 0 \le t \le 1, \qquad \text{and } y'_1(t) \le y'_2(t), \quad 0 \le t \le 1$$

- Assume

$$y_1(t) \le y_2(t), \qquad y'_1(t) \le y'_2(t),$$

then

$$Ky_1(t) \le Ky_2(t), \qquad (Ky_1)'(t) \le (Ky_2)'(t).$$

$$\int_0^1 G_1(t,s) f(s, y_1(s), y'_1(s)) ds \le \int_0^1 G_1(t,s) f(s, y_2(s), y'_2(s)) ds$$

$$\int_0^1 \frac{\partial}{\partial t} G_1(t,s) f\left(s, y_1(s), y'_1(s)\right) ds \le \int_0^1 \frac{\partial}{\partial t} G_1(t,s) f\left(s, y_2(s), y'_2(s)\right) ds$$

Since

$$\frac{d}{dt}\int_0^1 G_1(t,s) f ds = \int_0^1 \frac{\partial}{\partial t} G_1(t,s) f ds$$

- Assume existence of $w_0, v_0 \in C^2[0,1]$ satisfying:
  - $w_0(t) \le v_0(t),\ w'_0(t) \le v'_0(t),\ 0 \le t \le 1.$
  - $w_0(0) \le A \le v_0(0), w'_0(1) \le B \le v'_0(1).$
  - $w''_0(t) \ge f(t, w_0(t), w'_0(t)),\ 0 < t < 1.$
  - $v''_0(t) \le f(t, v_0(t), v'_0(t)),\ 0 < t < 1$

Define $w_{n+1} = Kw_n,\quad v_{n+1} = Kv_n$. Then

$$w_n(t) \le w_{n+1}(t) \le v_{n+1}(t) \le v_n(t), \qquad 0 \le t \le 1,$$

$$w'_n(t) \le w'_{n+1}(t) \le v'_{n+1}(t) \le v'_n(t) \quad 0 \le t \le 1,$$

Thus,

$$w_n \uparrow w \text{ in } C[0,1], \qquad w'_n \uparrow w' \text{ in } C[0,1],$$

$$v_n \downarrow v \text{ in } C[0,1], \qquad v'_n \downarrow v' \text{ in } C[0,1]$$

3. A Method of Forced Monoyonicity

Goal: Apply monotone methods to

$$y'' = f(t, y, y'), \qquad 0 < t < 1, \qquad y(0) = A, \quad y(1) = B, (*)$$

Since $\frac{\partial}{\partial t} G_0$ changes sign and f depends on y', the method of upper and lower solutions, coupled with monotone methods, has not been applied to (*). Consider an equivalent forced problem

$$y'' = f(t, y, y'), \qquad 0 < t < 1, \qquad y(0) = A, \quad y'(1) = g(y'(1), y(1)), (**)$$

$$Ky(t) = A + g(y'(1), y(1))t + \int_0^1 G_1(t,s) f(s, y(s), y'(s))\, ds$$

g is chosen so this forced boundary value problem is equivalent to (*) (or more precisely, if y is a solution of (**) then y is a solution of (*)).

If

$$\frac{\partial f}{\partial y} \le 0, \qquad \frac{\partial f}{\partial y'} \le 0,$$

and

$g(y_1, z_1) \le g(y_2, z_2)$ if $y_1 \le y_2$ and $z_1 \le z_2$, then K is monotone increasing and so the method of the upper and the lower solutions, coupled with monotone methods becomes a viable strategy.

Example

$$y'' = -2 + f(t, y, y'),\ 0 < t < 1,$$
$$y(0) = 0, \qquad y(1) = 3\ .(4)$$

Conjecture 1. If $f: [0,1] \times R^2 \to R$ is continuous,

$$\frac{\partial}{\partial y} f \le 0, \qquad \frac{\partial}{\partial y'} f \le 0 \quad \text{on } [0,1] \times R^2, f(t,0,0) = 0 \text{ and } |f(t, y_2, z_2) - f(t, y_1, z_1)| \le \epsilon_1 |y_2 - y_1| + \epsilon_2 |z_2 - z_1|$$

where $\epsilon_1, \epsilon_2 \ge 0$ and $\left(\frac{13}{6}\right)\epsilon_1 + 4\epsilon_2 \le \frac{3}{4}$. There exists a solution y of (4) satisfying

$$3t - t^2 \le y(t) \le 5t - t^2, \qquad 0 \le t \le 1,$$

$$3 - 2t \le y'(t) \le 5 - 2t, \qquad 0 \le t \le 1$$

- We must exhibit lower and upper solutions. We intend show that

$$w_0 = 3t - t^2, \qquad v_0 = 5t - t^2$$

are lower and upper solutions, respectively, for

$$y''(t) = -2 + f(t, y, y'), \qquad 0 < t < 1,$$
$$y(0) = 0, y'(1) = \frac{y'(1)}{y(1)} 3$$

Define $w_{n+1} = Kw_n,\ v_{n+1} = Kv_n$.

- $w_n \uparrow w$ in $C[0,1]$, $w'_n \uparrow w'$ in $C[0,1]$,
- $v_n \downarrow v$ in $C[0,1]$, $v'_n \downarrow v'$ in $C[0,1]$.

- Motivate the conjecture with $\epsilon_1, \epsilon_2 = 0$

$$y'' = -2, \qquad y(0) = 0, \qquad y(1) = 3$$

The solution is $y = 4t - t^2$, and the problem is equivalent to

$$y'' = -2,\ y(0) = 0,\ y'(1) = \frac{y'(1)}{y(1)} 3s$$

Construct $w_0(t) = 3t - t^2,\ v_0(t) = 5t - t^2$

$$w_1(t) = \frac{w'_0(1)}{w_0(1)} 3t + \int_0^1 G_1(t,s) w''_0(s) ds$$

$=\frac{3}{2}t+\int_0^1 G_1(t,s)(-2)ds=\frac{7}{2}t-t^2.$

$$v_1(t)=\frac{v'_0(1)}{v_0(1)}3t+\int_0^1 G_1(t,s)v''_0(s)ds$$

$=\frac{9}{4}t+\int_0^1 G_1(t,s)(-2)ds=\frac{17}{4}t-t^2.$

- We show directly that

$$w_0(t)\le w_1(t)\le v_1(t)\le v_0(t),\qquad 0\le t\le 1,$$

and

$$w'_0(t)\le w'_1(t)\le v'_1(t)\le v'_0(t),\quad 0\le t\le 1,$$

and then in theory

$$w_n\le w_{n+1}\le v_{n+1}\le v_n,\qquad 0\le t\le 1,$$

and

$$w'_n\le w'_{n+1}\le v'_{n+1}\le v'_n,\qquad 0\le t\le 1$$

To show these details directly,

$$w_0(t)=3t-t^2,\qquad w_1(t)=\frac{7}{2}t-t^2=\frac{5(3)-8}{3-1}t-t^2.$$

If

$$w_n(t)=a_nt-t^2,\qquad w'_n(t)=a_n-2t,$$

Then

$$w_{n+1}(t)=\frac{w'_n(1)}{w_n(1)}3t+2t-t^2=\frac{5a_n-8}{a_n-1}t-t^2=a_{n+1}t-t^2,$$

$$w'_{n+1}(t)=a_{n+1}-2t$$

Given $3\le a_n<4$, $a_{n+1}=\frac{5a_n-8}{a_n-1}$, show $a_n<a_{n+1}<4$.

- to obtain $a_{n+1}<4$, note that

$$2<a_n-1<3\Rightarrow\frac{-3}{a_n-1}<-1$$

Then,

$$a_{n+1}=\frac{5a_n-8}{a_n-1}=\frac{5(a_n-1)-3}{a_n-1}=5-\frac{3}{a_n-1}<5-1=4.$$

- to obtain $a_n<a_{n+1}$, note that

$$(a_n-2)(a_n-4)=a^2{}_n-6a_n+8<0,\quad for\ 3\le a_n<4$$

Then,

$$a^2{}_n-a_n<5a_n-8\text{ or }a_n<\frac{5a_n-8}{a_n-1}=a_{n+1}$$

Thus, $a_n\uparrow, a_n$ is bounded above by 4 so $a_n\to L$.

$a_{n+1}=\frac{5n-8}{a_n-1}\Longrightarrow L=\frac{5L-8}{L-1}$,

Then $L=2$ Or $L=4$

and so, $L=4$

- We want to show that

If $4<b_n\le 5$, $b_{n+1}<b_n<5$, then $\{b\}\downarrow 4t-t^2$.

- to obtain $4<b_{n+1}$

$$b_{n+1}=\frac{5a_n-8}{a_n-1}=\frac{5(a_n-1)-3}{a_n-1}=5-\frac{3}{a_n-1}>5-1=4$$

- to obtain $b_{n+1}<b_n$

Note: $(b_n-2)(b_n-4)=b^2{}_n-6b_n+8>0\ for\ 4<b_n\le 5$,

then $b^2{}_n-b_n>5b_n-8b_n(b_n)>5b_n-8$

or $b_n>\frac{5b_n-8}{b_n-1}=b_{n+1}$

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