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*Research Article*

# Common Fixed Point Theorem in Fuzzy Metric Space

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### *Abstract*

*In this paper, we prove a common fixed point theorem for six mappings which are weakly compatible and not necessary continuous mappings on fuzzy metric space*

***Keywords:*** *Common Fixed Point, Continuous mapping, Weakly Compatible Mapping.*

## 1. Introduction

[1]In 1965, the concept of fuzzy sets was first introduced by (Zadeh, 1965) in his classical paper. This theory evolved in many direction and application in wide verity of fields in which the phenomena under the studies are too complex. (Deng, 1982), (Erceg, 1988), (Kramosil and Michalek, 1975) have introduced the concept of fuzzy metric spaces in different ways. (Grabiec, 1998) followed (Kramosil and michalek ,1975) and obtained fuzzy version of (Banach's,1982) fixed point theorem. Banach fixed point theorem has many applications but suffer from one drawback the definition require that the mapping be the continuous throughout the space. Common fixed point theorem for commuting maps generalizing the (Banach's, 1982) fixed point theorem was proved by (Jungck, 1986) in 1976. Further (Jungck, 1998) more generalized commutatively, so called compatibility. There are proved many theorems in fuzzy metric space for compatible mappings. (Jungck and Rhoades, 1998) introduced the notion of the weakly compatible maps in 1998 and proved that compatible maps are weakly compatible maps but converse need not true. Here we prove result for fixed point theorem in fuzzy metric space by weakly compatible mappings.

## 2: Preliminaries

**Definition 1:** A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous $t-norm\ if\ *$ is satisfying the following conditions:
$(a)\ \ * \ is\ commutative\ and\ associative;$
$(b)\ * \ is\ continuous;$
$(c)\ \ a * 1 = a\ for\ all\ a \in [0,1];$
$(d)\ \ a * b \le c * d\ whenever\ a \le c$
$and\ b \le d\ and\ a, b, c, d \in [0,1].$

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**Definition 2:**

A $3-tuple(X, M,*)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuous t norm and $M$ is a fuzzy set on $X^2 \times (0.\infty)$ satisfying the following conditions; for all $x, y, z \in X\ and\ s, t > 0$,
$(a)\ \ M(x, y, t) > 0,$
$(b)\ \ M(x, y, t) = 1\ if\ and\ only\ if\ x = y$
$(c)\ \ M(x, y, t) =\ M(y, x, t),$
$(d)\ \ M(x, y, t) * M(y, z, s) \le\ M(x, z, t + s),$
$(e)\ M(x, y, t): (0, \infty) \to [0,1]\ is\ continuous.$
**Definition 3:** Let $(X, M,*)$ be a fuzzy metric space, then a sequence $\{x_n\}\ in\ X$ is said to be convergent to a point $x \in X$ if
$\mathrm{Lim}_{n\to}\ M(x_n, x, t) = 1, for\ all\ t > 0.$
**Definition 4:** Let $(X, M,*)$ be a fuzzy metric space, then a sequence $\{x_n\}\ in\ X$ is said to be Cauchy sequence if there exist a positive integer $n_0\ such\ that\ for\ m, n \ge n_0$
$\mathrm{Lim}_{n\to}\ M(x_m, x_n, t) = 1, for\ all\ t > 0.$
**Definition 5:** A fuzzy metric space $(X, M,*)$ is said to be complete fuzzy metric space if every Cauchy sequence converses in it.
**Definition 6:** Let $S\ and\ T$ be mappings from a fuzzy metric space $(X, M,*)$ into itself and $\{x_n\}$ is a sequence in $X$ such that
$\lim_{n\to\infty} Sx_n =\ \lim_{n\to\infty} Tx_n = z\ for\ some\ z \in X.$ Then the mappings $S\ and\ T$ are said to be compatible if
$\mathrm{Lim}_{n\to\infty}\ M(STx_n, TS\ x_n, t) = 1, for\ all\ t > 0.$
**Definition 7:** Two mappings $S\ and\ T$ are called weakly compatible in fuzzy metric space if they commute at their coincidence point; i.e. if $Su = Tu\ for\ some\ u \in X, then\ STu = TSu.$
**Lemma1:** $M(x, y, .)$ is non-decreasing function for all $x, y \in X.$
**Lemma2:** If $M(x, y, kt) \ge M(x, y, t) for\ all\ x, y \in X,$ $t \ge 0\ and\ for\ a\ number\ k \in (0,1)\ then\ \ x = y.$
**Definition 8:** Two self mappings $S\ and\ T$ of a fuzzy metric space $(X, M,*)$ are called non-compatible if there exist at least one sequence $\{x_n\}\ in\ X$ such that

$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \; for\; some\; z \in X$,but $\operatorname{Lim}_{n\to\infty} M(STx_n, TS\, x_n, t)$ is either not equal to 1 or non-existent. Sharma and Bamboria defined a property in the following way knows as (S-B) property.

**Definition 9:** Two self mappings $S\; and\; T$ of a fuzzy metric space $(X, M,*)$ satisfy the property (S-B) if there exist a sequence $\{x_n\}\; in\; X$ such that
$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z \; for\; some\; z \in X.$

## 3. Main Result

**Theorem:** Let $(X, M,*)$ be a fuzzy metric space with $t * t \geq t\; for\; all\; t \in [0,1]$ and the condition $\lim_{n\to\infty}(x, y, t) = 1,\quad for\; all\quad x, y \in X.$ Let $A, B, S, T, E\; and\; F$ are mappings from $X$ into itself such that

(1) $E(X) \subset AB(X)\; or\; F(X) \subset ST(X)$,
(2) $(AB, F)\; or\; (ST, E) satisfy\; the\; property\; (S - B)$,
(3) There exist a constant $k \in (0,1)$ such that
$M(Ex, Fy, kt) \geq M(ABy, Fy, t) * M(STx, Ex, t) * M(STx, Fy, t)$
$for\; all\; x, y \in X\;\; and\; t > 0$
(4) If one of $E(X), F(X), AB(X), ST(X)$ is a closed subset of $X$ then
(a) $F\; and\; AB$ have a coincidence point
(b) $E\; and\; ST$ have a coincidence point
Further if
(5) $FB = BF, ET = TE$,
(6) The pair $(AB, F) and\; (ST, E)$ are weakly compatible then
(c) $A, B, S, T, E, F$ have a unique common fixed point

**Proof:** Suppose that $(AB, F)$ satisfies the property $(S - B)$ then there exist a sequence $\{x_n\}\; in\; X$ such that
$\lim_{n\to\infty} Fx_n = \lim_{n\to\infty} ABx_n = z\; for\; some\; z \in X$
Since $F(X) \subset ST(X)$ there exist a sequence $\{y_n\}\; in\; X$ such that $Fx_n = STy_n$ .Hence $\lim_{n\to\infty} STy_n = \lim_{n\to\infty} Fx_n = z$. Now we show that $y_n = z$ .
Consider
$M(Ey_n, Fx_n, kt) \geq M(ABx_n, Fx_n, t) * M(STy_n, Ey_n, t) * M(STy_n, Fx_n, t)$
$\geq M(ABx_n, Fx_n, t) * M(Fx_n, Ey_n, t) * M(Fx_n, Fx_n, t)$
$\geq M(ABx_n, Fx_n, t) * M(Fx_n, Ey_n, t) * 1$

taking limit $n \to \infty$ and using the property $(S - B)$,we have

$\lim_{n\to\infty} M(Ey_n\, Fx_n, kt) \geq \lim_{n\to\infty} M(Fx_n, Ey_n, t)$
$\geq \lim_{n\to\infty} M(Ey_n\, Fx_n, t)$
then by theorem 2 we have
$\lim_{n\to\infty} Ey_n = \lim_{n\to\infty} Fx_n = z$
$\lim_{n\to\infty} Ey_n = z$

Now suppose that $ST(X)$ is closed then a subsequence of $\{y_n\}\; in\; X$ has a limit in $X$. Let it is $z$. Let $u = (ST)^{-1}z\;\; so\; STu = z$ . Now we have $\lim_{n\to\infty} Ey_n = \lim_{n\to\infty} ABx_n = \lim_{n\to\infty} STy_n = z$
Now
$M(Eu, Fx_n, kt) \geq M(ABx_n, Fx_n, t) * M(STu, Eu, t) * M(STu, Fx_n, t)$

taking limit $n \to \infty$ and using the property $(S - B)$, we have
$\lim_{n\to\infty} M(Eu, Fx_n, kt) \geq 1 * M(z, Eu, t) * 1$
$M(Eu, z, kt) \geq M(z, Eu, t)$
$\geq M(Eu, z, t)$

then by lemma (2) we have $Eu = z$. This shows u is coincidence point of mappings E and ST, this proves (a).

Since $E(X) \subset AB(X) and\; Eu = z \implies z \in AB(X)$.
$Let\; v \in (AB)^{-1}z. Then\; z = ABv$

Now consider
$M(Ey_n, Fv, kt) \geq M(ABv, Fv, t) * M(STy_n, Ey_n, t) * M(STy_n, Fv, t)$
$\geq M(z, Fv, t) * M(STy_n, Ey_n, t) * M(STy_n, Fv, t)$

taking limit $n \to \infty$ and using the property $(S - B)$, we have
$M(z, Fv, kt) \geq M(z, Fv, t) * 1 * M(z, Fv, t)$
$\geq M(z, Fv, t)$

then by lemma (2) we have $Fv = z$.This shows that v is a coincidence point of mappings F and AB, this proves (b). Similarly we can proves if $AB(X)$ is closed. Now let if $E(X) or\; F(X)$ is closed then by condition (1), $z \in E(X) \subset AB(X) or\; z \in F(X) \subset ST(X)$ respectively, then (a) and (b) are completely established.

We are given the pair $(ST, E)$ is weakly compatible so ST and E commute at their coincidence point i.e. $ST(Eu) = E(STu)\; or\; STz = Ez$. Similarly $AB(Fv) = F(ABv)\; or\; ABz = Fz$.Now we have to prove that $Ez = z$.From (3) we have
$M(Ez, Fx_n, kt) \geq M(ABx_n, Fx_n, t) * M(STz, Ez, t) * M(STz, Fx_n, t)$
$\geq M(ABx_n, Fx_n, t) * 1 * M(STz, Fx_n, t)$

taking limit $n \to \infty$,we have
$M(Ez, z, kt) \geq 1 * M(Ez, z, t)$
Then by lemma (2), we have
$Ez = z. Since\; STz = Ez\;\; so\; we\; have$
$Ez = STz = z$
Now we have to show that $Fz = z$. From (3) we have

$M(Ey_n, Fz, kt) = M(Ez, Fz, kt)$
$\geq M(ABz, Fz, t) * M(STy_n, Ey_n, t) * M(STy_n, Fz, t)$
$1 * M(STy_n, Ey_n, t) * M(STy_n, Fz, t)$

taking limit $n \to \infty$,we have
$M(z, Fz, kt) \geq M(z, F, t)$

Then by lemma (2), we have $Fz = z. Since\; ABz = Fz\;\; so\; we\; have\; Fz = ABz = z.$
Now we show that $Bz = z$. Consider
$M(z, Bz, kt) = M(Ez, B(Fz), kt)$
$since\; Ez = Fz = z$
$= M(Ez, F(Bz), kt)$
$\geq M(AB(Bz), F(Bz), t) *$

$M(STz, Ez, t) * M(STz, F(Bz), t)$
$= M(B(ABz), B(Fz), t) * 1 * M(z, B(Fz), t)$
$\geq M(z, Bz, t)$

Then by lemma (2) we have $Bz = z$ . $since\ ABz = z\ shows\ Az = z$. Finally we have to show that $Tz = z$.Consider
$M(z, Tz, kt) = M(Ez, T(Fz), kt)$
$= M(Ez, F(Tz), kt)$
$\geq M(AB(Tz), F(Tz), t) * M(STz, Ez, t)$
$* M(STz, F(Tz), t)$
$\geq M(Tz, Tz, t) * 1 * M(z, Tz, t)$
$\geq M(z, Tz, t)$

Then by lemma (2) we have $Tz = z$ . $since\ STz = z\ shows\ Sz = z$.From above we have $Az = Bz = Sz = Tz = Fz = Ez = z$.
Let w be another common fixed point of mappings $A, B, S, T, E, F. Consider$
$M(z, w, kt) = M(Ez, Fw, kt)$
$\geq M(ABw, Fw, t) * M(STz, Ez, t) * M(STz, Fw, t)$
$\geq 1 * 1 * M(z, w, t)$
$\geq M(z, w, t)$
Then by lemma (2) we have $w = z$.

**Conclusion**

In this paper, we prove a common fixed point theorem for six mappings which are weakly compatible and not necessary continuous mappings on fuzzy metric space

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