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Research Article

Dynamic Analysis of Laminated Sandwich Beams using Various Displacement Field Forms and Various Boundary Conditions

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Abstract

The present paper deals with the analysis of displacement field forms effects on natural frequencies for flexure problems of laminated sandwich beams. Several forms using various parameters are tested. Both analytical and finite elements formulations using Hamilton's principle are carried out. Numerical results have been computed of a sandwich beam and compared in order to highlight the importance of inclusion of such parameters and its capacity for good estimation of natural frequencies of sandwich beams.

Keywords: *Sandwich beams, Free Vibration, Displacement field forms, Natural frequencies.*

1. Introduction

Sandwich structures (Berthelot, 1992), (Reddy, 1997) have been regarded as a convenient strategy for many industries as aerospace, automobile, nuclear, marine, biomedical and civil engineering. This is due to its high strength and high stiffness to weight ratio, good resistance to fatigue and corrosion phenomenon.

Since these structures are made of two or more layered materials, their manufacturing cost is superior to traditional materials. Nevertheless, their advantages make them an efficient solution for such manufactories especially in the aircraft industry when the safety of the aircraft is an important design factor. Hence, it is necessary to analyze their macro-mechanical characteristics such as deflections, stress and strain distribution through the thickness, natural frequencies, modal deformations, and the effect of boundary conditions and external loads. For that, an efficient theory is required for accurate prediction of the structural characteristics of these beams.

Several researchers (Meunier and Shenoi, 2001), (Boubaker, *et al*, 2002), (Chandra, *et al*, 2002), (Ghugal and Shimpi, 2002), (Meunier and Shenoi, 2003), (Rathbun, *et al*, 2006), (Soula, *et al*, 2006), (Bilasse, *et al*, 2010), (Jian, *et al*, 2014) have investigated in the study of laminated sandwich beams. One of the well known theories is the classical theory developed by Euler-Bernoulli (Zienkiewciz and Taylor, 2000) which is useful only for thin beams because it has neglected transverse shear deformations. It is obvious that transverse shear deformations have to be taken into account in the analysis. Thus, Timoshenko (Timoshenko, 1922) has developed a beam theory to include this effect. The theory assumes a constant shear strain across the thickness of the beam and requires a shear correction factor. Following his work, many theories incorporating the effect of shear deformation have been developed (Mindlin and Goodman, 1950), (Cowper, 1968), (Levinson, 1981), (Banerjee, 2001), (Banerjee, 2004), (Nilsson E and Nilsson AC, 2002).Then, some authors (Bickford, 1982), (Heyliger and Reddy, 1988), (Degiovani, *et al*., 2010), (Carrera, *et al*., 2011), (Damanpack and Khalili, 2012) have developed a high order beam theory. While these theories do not require a shear correction factor as Timoshenko theory, the resulting differential equation is sixth order whereas that for a consistent beam theory is of the fourth-order. Furthermore, the variation of boundary conditions is not studied. Therefore, the form of the displacement field which will be used must considers some parameters, to overcome the following drawbacks presented in previous works: neglecting the shear deformations, sixth order of differential equation which complicates further the analytical solutions, assuming a constant variation of the shear deformation across the thickness of the beams or using only one boundary condition.

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The present paper proposes to analyze the effect of the inclusion of some parameters in the displacement field forms on the prediction of dynamic behavior of sandwich beams using various boundary conditions. Firstly, equations of motion are derived for each form of displacements field using Hamilton's principle. Then analytical solutions as well as finite elements (FEM) solutions are established. Numerical simulations of laminated sandwich beam using various forms of displacements field and various boundary conditions are illustrated for the prediction of natural frequencies. A comparative study is also illustrated to perform the importance of inclusion of some parameters in the displacements form.

2. Formulation

Multilayer structures are typically used for its light-weight, high specific stiffness and strength values in many engineering fields. In fact, there are attempts to replace components with classical materials (steel, concrete) by laminated materials notably sandwich structures. The studied sandwich panel is constituted by three layers: two elastic faces and a homogeneous honeycomb core. It is assumed to have a length L, width b and total thickness *2h* as shown in Fig.1.

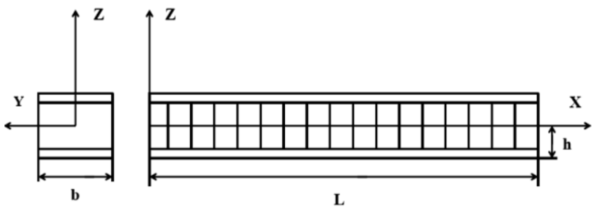

Fig.1 Geometry and coordinate system of the sandwich beam

Various forms of the displacement field, as well as, boundary conditions will be studied to evaluate the effect of the parameters included in the displacement forms on the dynamic behavior of the sandwich beams.

2.1. Hypothesis

It is assumed that the study domain is linear elastic with small displacements, the length of the beam is quite large compared to others dimensions (beams theory), the faces and the Honeycomb core materials are isotropic homogeneous. Furthermore, the continuity of displacements along the interfaces between the layers is considered. No slip or delamination between the layers.

2.2. Displacement field form without shear and without warping effects (SCG)

The displacement field of a sandwich beam without taking account the shear and warping effects can be expressed as follows:

$$U(M,t)=\begin{Bmatrix} u(x,y,z,t)=u_0(x,t)-z\dfrac{\partial w_b(x,t)}{\partial x} \\ w(x,y,z,t)=w_b(x,t) \end{Bmatrix} \quad (1)$$

Where: $u_0(x,t)$ is the displacement due to extension and $w_b(x,t)$ is the displacement due to bending deformations measured at the mid-surface of the sandwich beam (Z=0).

The corresponding relation strain-displacement for this form of displacement (Eq.1) is given as follows:

$$\varepsilon=\begin{Bmatrix} \varepsilon_{xx}=\dfrac{\partial u_0}{\partial x}-z\dfrac{\partial^2 w_b}{\partial x} \\ \varepsilon_{zz}=0 \\ \gamma_{xz}=0 \end{Bmatrix} \quad (2)$$

The strain field in the case where the shear and warping effects are neglected can be written as follows:

$$\varepsilon_{xx}=\varepsilon_1^0+z\varepsilon_1^1 \quad (3)$$

Where: $\varepsilon_1^0=\dfrac{\partial u_0}{\partial x}$ and $\varepsilon_1^1=-\dfrac{\partial^2 w_b}{\partial x}$ are respectively the membrane and bending strains contribution.

The stress-strain relationship of the kth layer is expressed as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{zz} \\ \tau_{xz} \end{Bmatrix}^{(k)}=\begin{bmatrix} Q_{11} & Q_{13} & 0 \\ Q_{13} & Q_{33} & 0 \\ 0 & 0 & Q_{44} \end{bmatrix}\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{zz} \\ \gamma_{xz} \end{Bmatrix}^{(k)} \quad (4)$$

Or in compact form:

$$\{\sigma\}=[Q]\{\varepsilon\} \quad (5)$$

Where: $[Q]$ is the reduced elastic stiffness matrix which contains the elastic materials constants defined in the orthotropic axis as follows:

$$Q_{11}=\frac{E_1}{1-\upsilon_{12}\upsilon_{21}};\ Q_{13}=\upsilon_{21}\frac{E_1}{1-\upsilon_{12}\upsilon_{21}};\ Q_{33}=\frac{E_2}{1-\upsilon_{12}\upsilon_{21}}$$
$$Q_{44}=G_{c12} \quad (6)$$

Where: E_1 ; E_2 ; υ_{12} ; υ_{21} and G_{c12} are the engineers constants in the orthotropic axis of the kth layer of the corresponding sandwich beam (Young modulus, Poisson ratio, shear modulus).

The variational Hamilton's principle which is based on the calculation of the variation of kinetic and potential energies is applied to derive the equations of motion. Hence, this principle can be expressed as follows:

$$\int_0^t \delta\left(E_c-E_p\right)=0 \quad (7)$$

The kinetic energy is given as:

$$E_c = \frac{1}{2}\int_V \rho\left(\dot{u}^2 + \dot{w}^2\right)dV \tag{8}$$

$\dot{u}$ and $\dot{w}$ are the time derivatives of *u* and *w;* ρ is the density of the corresponding material.
The potential energy is given as:

$$E_p = \frac{1}{2}\int_V \left(\sigma_{xx}\delta\varepsilon_{xx} + \sigma_{zz}\delta\varepsilon_{zz} + \tau_{xz}\delta\gamma_{xz}\right)dV \tag{9}$$

Since $\varepsilon_{zz} = \gamma_{xz} = 0$, the kinetic and potential energy variations are given as:

$$\int_0^t \delta E_c dt = \rho\int_0^t\int_V \left(\dot{u}\delta u + \dot{w}\delta w\right)dVdt \tag{10}$$

$$\int_0^t \delta E_p dt = \int_0^t\int_V \left(\sigma_{xx}\delta\varepsilon_{xx}\right)dVdt \tag{11}$$

Substituting (Eq.10) and (Eq.11) by its expressions, the Hamilton's principle (Eq.7) can be rewritten as follows:

$$\int_0^t\int_V \begin{bmatrix} \left(\sigma_{xx}\delta\varepsilon_{xx}\right) - \rho\left\{\frac{\partial u_0}{\partial t} - z\frac{\partial^2 w_b}{\partial x\partial t}\right\}\left\{\frac{\partial \delta u_0}{\partial t} - z\frac{\partial^2 \delta w_b}{\partial x\partial t}\right\} \\ -\rho\left\{\frac{\partial w_b}{\partial t}\right\}\left\{\frac{\partial \delta w_b}{\partial t}\right\} \end{bmatrix} dVdt = 0 \tag{12}$$

Integrating the appropriate terms in (Eq.12) by parts and collecting the coefficients of δu_0 and δw_b, the equations of motion in terms of stress resultants are expressed as:

$$\frac{\partial N}{\partial x} - I_0\frac{\partial^2 u_0}{\partial t} + I_1\frac{\partial^3 w_b}{\partial x\partial t^2} = 0 \tag{13a}$$

$$\frac{\partial^2 M}{\partial^2 x} - I_0\frac{\partial^2 w_b}{\partial t^2} - I_1\frac{\partial^3 u_0}{\partial x\partial t^2} + I_2\frac{\partial^4 w_b}{\partial x^2\partial t^2} = 0 \tag{13b}$$

The laminate stiffness constants, the stress resultants and the mass moments of inertia are defined, respectively, as follows:

$$\left(I_0, I_1, I_2\right) = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} \rho\left(1, z, z^2\right)dz \tag{14a}$$

$$\left(N, M\right) = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} \sigma_{xx}\left(1, z\right)dz \tag{14b}$$

$$\left(A_{11}, A_{12}, B_{11}\right) = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} Q_{11}^i\left(1, z, z^2\right)dz \tag{14c}$$

N_c is the number of layers of the sandwich beam.
Hence, the derived equations of motion in term of displacements are given as follows:

$$\begin{Bmatrix} A_{11}\frac{\partial^2 u_0}{\partial x^2} - A_{12}\frac{\partial^3 w_b}{\partial x^3} - I_0\frac{\partial^2 u_0}{\partial t} + I_1\frac{\partial^3 w_b}{\partial x\partial t^2} = 0 \\ A_{12}\frac{\partial^3 u_0}{\partial x^3} - B_{11}\frac{\partial^4 w_b}{\partial x^4} - I_0\frac{\partial^2 w_b}{\partial t^2} - I_1\frac{\partial^3 u_0}{\partial x\partial t^2} + I_2\frac{\partial^4 w_b}{\partial x^2\partial t^2} = 0 \end{Bmatrix} \tag{15}$$

2.2.1. Analytical Solutions

The analytical solutions of the equations of motion (Eq.15) are given by assuming that:

$$u_0\left(x,t\right) = A\left(x\right)e^{j\omega t}\text{; } w_b\left(x,t\right) = B\left(x\right)e^{j\omega t} \tag{16}$$

Where ω is the natural frequency of the sandwich beam. Substituting (Eq.16) into (Eq.15), the following equations are obtained:

$$\begin{Bmatrix} A_{11}A^{(2)}\left(x\right) - A_{12}B^{(3)}\left(x\right) - I_0 A\left(x\right) \\ +I_1\omega^2 B^{(1)}\left(x\right) = 0 \\ A_{12}A^{(4)}\left(x\right) - B_{11}B^{(4)}\left(x\right) + I_0\omega^2 B\left(x\right) + I_1\omega^2 A^{(1)}\left(x\right) \\ -I_2\omega^2 B^{(2)}\left(x\right) = 0 \end{Bmatrix} \tag{17}$$

Where the subscript (i) i=1,...,4 indicates the derivative order of the equations of motion.
After assuming that $A\left(x\right) = A_0 e^{rx}$ and $B\left(x\right) = B_0 e^{rx}$, the analytical solutions of the equation of motion system (Eq.17) can be taken as:

$$A\left(x\right) = A_{01}e^{ja\beta x} + A_{02}e^{-ja\beta x} \tag{18a}$$

$$B\left(x\right) = B_{01}e^{\sqrt{\beta}x} + B_{02}e^{-\sqrt{\beta}x} + B_{03}e^{j\sqrt{\beta}x} + B_{04}e^{-j\sqrt{\beta}x} \tag{18b}$$

Where: $\beta = \sqrt{\frac{I_0\omega^2}{B_{11}}}$ and $a = \sqrt{\frac{B_{11}}{A_{11}}}$.

The application of various boundary conditions as simply-supported, clamped-free and bi-clamped to the sandwich beam at x=0 and x=L can be summarized for each type in Table 1.

Table1 Boundary conditions without shear and without warping effects

Boundary Condition	Associated Equation in matrix form
Simply Supported	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & -1 & -1 \\ 0 & 0 & 1 & -e^{-2\sqrt{\beta}L} & e^{(j-1)\sqrt{\beta}L} & e^{-(j+1)\sqrt{\beta}L} \\ 0 & 0 & 1 & e^{-2\sqrt{\beta}L} & -e^{(j-1)\sqrt{\beta}L} & -e^{-(j+1)\sqrt{\beta}L} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -e^{2ja\beta L} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_{01} \\ A_{02} \\ B_{01} \\ B_{02} \\ B_{03} \\ B_{04} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Clamped Free	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & j & -j \\ 0 & 0 & 1 & -e^{-2\sqrt{\beta}L} & -je^{(j-1)\sqrt{\beta}L} & je^{-(j+1)\sqrt{\beta}L} \\ 0 & 0 & 1 & e^{-2\sqrt{\beta}L} & -e^{(j-1)\sqrt{\beta}L} & -e^{-(j+1)\sqrt{\beta}L} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -e^{-2ja\beta L} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_{01} \\ A_{02} \\ B_{01} \\ B_{02} \\ B_{03} \\ B_{04} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Clamped Clamped	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 & j & -j \\ 0 & 0 & 1 & -e^{-2\sqrt{\beta}L} & je^{(j-1)\sqrt{\beta}L} & -je^{-(j+1)\sqrt{\beta}L} \\ 0 & 0 & 1 & e^{-2\sqrt{\beta}L} & e^{(j-1)\sqrt{\beta}L} & e^{-(j+1)\sqrt{\beta}L} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -e^{-2ja\beta L} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_{01} \\ A_{02} \\ B_{01} \\ B_{02} \\ B_{03} \\ B_{04} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

The resolution of the associated equation in matrix form for each type of applied boundary condition enables to determine the natural frequencies of the sandwich beam using Newton-Raphson procedure.

2.2.2. Finite Elements Solutions

In this section, the finite element procedure for the adopted displacement field is developed. Both elements have the same number of degree-of-freedom (dofs) per node, each element having two nodes and each node having three degree of freedom. Linear polynomials are used for nodal variable u_0 as well as Hermite cubic polynomials are used for the other variables of the elements. The displacements field given by (Eq.1) is rewritten in the matrix form as:

$$\{U\}=[Z]\{d\} \quad (19)$$

Where:

$$\{U\}=\{u \quad w\}^T \quad (20)$$

$$[Z]=\begin{bmatrix}1 & 0 & -z\\ 0 & 1 & 0\end{bmatrix} \quad (21)$$

$$\{d\}=\left\{u_0 \quad w_b \quad \frac{\partial w_b}{\partial x}\right\}^T \quad (22)$$

The strain field associated to (Eq.19) is given as follows:

$$\{\varepsilon\}=[Z_i]\{k\} \quad (23)$$

Where: $\{\varepsilon\}=\{\varepsilon_{xx} \quad \varepsilon_{zz} \quad \gamma_{xz}\}^T$; $[Z_i]=\begin{bmatrix}1 & z\\ 0 & 0\\ 0 & 0\end{bmatrix}$;

$\{k\}=\{\varepsilon_1^0 \quad \varepsilon_1^1\}^T$.

Applying the variational Hamilton Principal, the variation of potential energy becomes:

$$\begin{aligned}\int_0^t \delta E_p dt &= \int_0^t\int_V \{\delta\varepsilon\}^T\{\sigma\}\, dVdt \\ &= b\int_0^t\int_0^x\int_{-h}^{h}\{\delta k\}^T[Z_i]^T[\sigma]\,dzdAdt \\ &= b\int_0^t\int_0^x\{\delta k\}^T\left(\int_{-h}^{h}[Z_i]^T\{\sigma\}\,dz\right)dxdt \\ &= b\int_0^t\int_0^x\{\delta k\}^T\{S\}dxdt\end{aligned} \quad (24)$$

Where the stress resultants: $\{S\}=\{N \quad M\}$

Hence: $\{S\}=[D]\{k\}$

With: $[D]=\sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}}[Z_i]^T[Q][Z_i]dz$

Substituting the matrix $\{S\}$ by its expression into (Eq.24) leads to:

$$\int_0^t \delta E_p dt = b\int_0^t\int_0^x \{\delta k\}^T[D]\{k\}\,dxdt \quad (25)$$

The variation of kinetic energy is given as:

$$\begin{aligned}\int_0^t \delta E_c dt &= \rho\int_0^t\int_V(\dot u\delta u+\dot w\delta w)dVdt \\ &= -\rho\int_0^t\int_V\{\delta U\}^T\{\ddot U\}\,dVdt \\ &= -\rho\int_0^t\int_V\{\delta d\}^T[Z]^T[Z]\{\ddot d\}\,dVdt \\ &= -\rho b\int_0^t\int_0^x\{\delta d\}^T\left(\int_{z_i}^{z_{i+1}}[Z]^T[Z]\,dz\right)\{\ddot d\}\,dxdt \\ &= -b\int_0^t\int_0^x\{\delta d\}^T[I_m]\{\ddot d\}\,dxdt\end{aligned} \quad (26)$$

With the mass moment of inertia matrix: $[I_m]=\int_{z_i}^{z_{i+1}}\rho[Z]^T[Z]dz$

Then the corresponding global matrices are assembled accounting for the connectivity using the standard assembling procedure and the following equation of motion is established:

$$[M]\{\ddot U\}+[K]\{U\}=0 \quad (27)$$

2.3. Displacement field form with shear and without warping effects (ACSG)

In this section, analytical and finite elements (FEM) formulations are indicated briefly. In fact, the development steps of (FEM) method are the same as presented in the previous section (2.2.2). Hence, the displacement field form with shear and without warping effects can be written as:

$$U(M,t)=\begin{Bmatrix} u(x,y,z,t)=u_0(x,t)-z\dfrac{\partial w_b(x,t)}{\partial x} \\ w(x,y,z,t)=w_b(x,t)+\boldsymbol{w_0(x,t)}\end{Bmatrix} \quad (28)$$

Where: $\boldsymbol{w_0(x,t)}$ is the displacement due to shear deformations contribution.

The corresponding strain field is expressed as follows:

$$\varepsilon=\begin{Bmatrix}\varepsilon_{xx}=\dfrac{\partial u_0}{\partial x}-z\dfrac{\partial^2 w_b}{\partial x}=\varepsilon_2^0+z\varepsilon_2^1 \\ \varepsilon_{zz}=0 \\ \gamma_{xz}=\dfrac{\partial w_0}{\partial x}=\gamma_3^0\end{Bmatrix} \quad (29)$$

Where: γ_3^0 represents the shear deformations effect.

After the application of the Hamilton' principle, the derived equation of motion in terms of δu_0; δw_b and δw_0 are given as:

$$\begin{Bmatrix} A_{11}\dfrac{\partial^2 u_0}{\partial x^2} - A_{12}\dfrac{\partial^3 w_b}{\partial x^3} - I_0\dfrac{\partial^2 u_0}{\partial t} + I_1\dfrac{\partial^3 w_b}{\partial x \partial t^2} = 0 \\ A_{12}\dfrac{\partial^3 u_0}{\partial x^3} - B_{11}\dfrac{\partial^4 w_b}{\partial x^4} - I_0\dfrac{\partial^2 w_b}{\partial t^2} \\ -I_1\dfrac{\partial^3 u_0}{\partial x \partial t^2} + I_2\dfrac{\partial^4 w_b}{\partial x^2 \partial t^2} - I_0\dfrac{\partial^2 w_0}{\partial t^2} = 0 \\ G_{11}\dfrac{\partial^2 w_0}{\partial x^2} - I_0\dfrac{\partial^2 w_b}{\partial t^2} - I_0\dfrac{\partial^2 w_0}{\partial t^2} = 0 \end{Bmatrix} \tag{30}$$

Where $\left(I_0, I_1, I_2\right)$ have the same expressions as defined in section (2.2). The additional stress resultant and laminated stiffness constant due to shear contribution are defined as follows:

$$Q_1 = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} \sigma_{xz}dz \text{ and } G_{11} = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} Q_{44}^{i}dz .$$

Hence, the analytical solutions can be expressed as follows:

$$A_c\left(x\right) = A_{0c1}e^{r_{1c}x} + A_{0c2}e^{r_{2c}x} \tag{31a}$$

$$B_c\left(x\right) = B_{0c1}e^{r_{3c}x} + B_{0c2}e^{r_{4c}x} + B_{0c3}e^{r_{5c}x} + B_{0c4}e^{r_{6c}x} \tag{31b}$$

Where:

$$r_{1c} = \frac{\sqrt{-A_{11}I_0}\,\omega}{A_{11}};\ r_{2c} = -\frac{\sqrt{-A_{11}I_0}\,\omega}{A_{11}};$$

$$r_{3c} = r_{5c} = \frac{1}{2}\frac{\sqrt{-2B_{11}G_{11}\left(B_{11}I_0\omega - \sqrt{B_{11}^2 I_0^2\omega^2 + 4B_{11}G_{11}^2 I_0}\right)\omega}}{B_{11}G_{11}}$$

$$r_{4c} = r_{6c} = -\frac{1}{2}\frac{\sqrt{-2B_{11}G_{11}\left(B_{11}I_0\omega - \sqrt{B_{11}^2 I_0^2\omega^2 + 4B_{11}G_{11}^2 I_0}\right)\omega}}{B_{11}G_{11}}$$

The boundary conditions formulation is illustrated in Table2.

Table2 Boundary conditions with shear and without warping effects

Boundary Condition	Associated Equation in matrix form
Simply Supported	$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & r_{3c}^2 & r_{4c}^2 & r_{5c}^2 & r_{6c}^2 \\ 0 & 0 & e^{r_{3c}L} & e^{r_{4c}L} & e^{r_{5c}L} & e^{r_{6c}L} \\ 0 & 0 & r_{3c}^2 e^{r_{3c}L} & r_{4c}^2 e^{r_{4c}L} & r_{5c}^2 e^{r_{5c}L} & r_{6c}^2 e^{r_{6c}L} \\ 1 & 1 & 0 & 0 & 0 & 0 \\ r_{1c}e^{r_{1c}L} & r_{2c}e^{r_{2c}L} & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} A_{0c1} \\ A_{0c2} \\ B_{0c1} \\ B_{0c2} \\ B_{0c3} \\ B_{0c4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Clamped Free	$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & r_{3c} & r_{4c} & r_{5c} & r_{6c} \\ r_{1c}e^{r_{1c}L} & r_{2c}e^{r_{2c}L} & 0 & 0 & 0 & 0 \\ 1 & 1 & r_{3c}^2 e^{r_{3c}L} & r_{4c}^2 e^{r_{4c}L} & r_{5c}^2 e^{r_{5c}L} & r_{6c}^2 e^{r_{6c}L} \\ 0 & 0 & r_{3c}^2 e^{r_{3c}L} & r_{4c}^2 e^{r_{4c}L} & r_{5c}^2 e^{r_{5c}L} & r_{6c}^2 e^{r_{6c}L} \end{bmatrix} \begin{Bmatrix} A_{0c1} \\ A_{0c2} \\ B_{0c1} \\ B_{0c2} \\ B_{0c3} \\ B_{0c4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$
Clamped Clamped	$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & r_{3c} & r_{4c} & r_{5c} & r_{6c} \\ e^{r_{1c}L} & e^{r_{2c}L} & 0 & 0 & 0 & 0 \\ 1 & 1 & e^{r_{3c}L} & e^{r_{4c}L} & e^{r_{5c}L} & e^{r_{6c}L} \\ 0 & 0 & r_{3c}e^{r_{3c}L} & r_{4c}e^{r_{4c}L} & r_{5c}e^{r_{5c}L} & r_{6c}e^{r_{6c}L} \end{bmatrix} \begin{Bmatrix} A_{0c1} \\ A_{0c2} \\ B_{0c1} \\ B_{0c2} \\ B_{0c3} \\ B_{0c4} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

The natural frequencies associated to this form are determined by applying Newton-Raphson procedure. The derivation of the equation of motion using the finite elements procedure is obtained by substituting $\left[Z\right]$ and $\left[Z_i\right]$ by their new expressions as:

$$\left[Z\right] = \begin{bmatrix} 1 & 0 & -z & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } \left[Z_i\right] = \begin{bmatrix} 1 & z & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

2.3. Displacement field form with shear and with warping effects (ACG)

In this section, both shear and warping effects are considered. Hence, the displacements field can be expressed as follows:

$$U\left(M,t\right) = \begin{Bmatrix} u\left(x,y,z,t\right) = u_0\left(x,t\right) - z\dfrac{\partial w_b\left(x,t\right)}{\partial x} \\ -\dfrac{\boldsymbol{z^3}}{\boldsymbol{3h^2}}\dfrac{\partial \boldsymbol{w_0}\left(\boldsymbol{x,t}\right)}{\partial \boldsymbol{x}} \\ w\left(x,y,z,t\right) = w_b\left(x,t\right) + \boldsymbol{w_0}\left(\boldsymbol{x,t}\right) \end{Bmatrix} \tag{32}$$

The warping effect contribution is introduced by the term $\dfrac{\boldsymbol{z^3}}{\boldsymbol{3h^2}}\dfrac{\partial \boldsymbol{w_0}\left(\boldsymbol{x,t}\right)}{\partial \boldsymbol{x}}$ which indicates a cubic variation of the displacements field through the thickness.

The strain field is expressed as follows:

$$\varepsilon = \begin{Bmatrix} \varepsilon_{xx} = \dfrac{\partial u_0}{\partial x} - z\dfrac{\partial^2 w_b}{\partial x} - \dfrac{z^3}{3h^2}\dfrac{\partial^2 w_0}{\partial x} = \varepsilon_3^0 + z\varepsilon_3^1 + z^3\varepsilon_3^2 \\ \varepsilon_{zz} = 0 \\ \gamma_{xz} = \left(1 - \dfrac{z^2}{h^2}\right)\dfrac{\partial w_0}{\partial x} = \gamma_4^0 + z^2\gamma_4^1 \end{Bmatrix} \tag{33}$$

The obtained equations of motion are given as:

$$\begin{Bmatrix} A_{11}\dfrac{\partial^2 u_0}{\partial x^2} - A_{12}\dfrac{\partial^3 w_b}{\partial x^3} - \dfrac{A_{13}}{3h^2}\dfrac{\partial^3 w_0}{\partial x^3} - I_0\dfrac{\partial^2 u_0}{\partial t} \\ +I_1\dfrac{\partial^3 w_b}{\partial x \partial t^2} + \dfrac{I_3}{3h^2}\dfrac{\partial^3 w_0}{\partial x \partial t^2} = 0 \\ A_{12}\dfrac{\partial^3 u_0}{\partial x^3} - B_{11}\dfrac{\partial^4 w_b}{\partial x^4} - \dfrac{B_{12}}{3h^2}\dfrac{\partial^3 w_0}{\partial x^3} \\ -I_0\dfrac{\partial^2 w_b}{\partial t^2} - I_1\dfrac{\partial^3 u_0}{\partial x \partial t^2} + \dfrac{I_4}{3h^2}\dfrac{\partial^4 w_0}{\partial x^2 \partial t^2} - I_0\dfrac{\partial^2 w_0}{\partial t^2} = 0 \\ \dfrac{A_{13}}{3h^2}\dfrac{\partial^3 u_0}{\partial x^3} - \dfrac{B_{12}}{3h^2}\dfrac{\partial^4 w_b}{\partial x^4} - \dfrac{B_{13}}{9h^2}\dfrac{\partial^4 w_0}{\partial x^4} \\ +\left(G_{11} - \dfrac{2G_{12}}{h^2} - \dfrac{2G_{14}}{h^4}\right)\dfrac{\partial^2 w_0}{\partial x^2} - \dfrac{I_3}{3h^2}\dfrac{\partial^3 u_0}{\partial x \partial t^2} \\ +\dfrac{I_6}{9h^2}\dfrac{\partial^4 w_0}{\partial x^2 \partial t^2} - I_0\dfrac{\partial^2 w_b}{\partial t^2} - I_0\dfrac{\partial^2 w_0}{\partial t^2} = 0 \end{Bmatrix} \tag{34}$$

The additionally terms due to inclusion of warping effect are defined as:

$$\left(I_3, I_4, I_6\right) = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} \rho\left(z^3, z^4, z^6\right)dz \tag{35a}$$

$$\left(A_{13}, B_{12}, B_{13}\right) = \sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}} Q_{11}^{i}\left(z^3, z^4, z^6\right)dz \tag{35b}$$

$$\left(G_{12},G_{14}\right)=\sum_{i=1}^{Nc}\int_{z_i}^{z_{i+1}}Q_{44}^{i}\left(z^2,z^4\right)dz \tag{35c}$$

The analytical solutions have, here, the following form:

$$A_g\left(x\right)=A_{0g1}e^{r_{1g}x}+A_{0g2}e^{r_{2g}x} \tag{36a}$$

$$B_g\left(x\right)=B_{0g1}e^{r_{3g}x}+B_{0g2}e^{r_{4g}x}+B_{0g3}e^{r_{5g}x}+B_{0g4}e^{r_{6g}x} \tag{36b}$$

Where: $r_{1g}=\dfrac{\sqrt{-A_{11}I_0}\,\omega}{A_{11}}$; $r_{2g}=-\dfrac{\sqrt{-A_{11}I_0}\,\omega}{A_{11}}$

$$r_{5g}=r_{3g}=\frac{1}{2B_{11}\left(G_{11}h^2-2G_{12}\right)}\left(\begin{array}{l}-2B_{11}\left(G_{11}h^2-2G_{12}\right)\times\\ \left(I_0B_{11}h^2\omega-\sqrt{\begin{array}{l}I_0B_{11}^2h^4\omega^2+4B_{11}G_{11}^2h^4I_0\\-16B_{11}G_{11}h^2I_0G_{12}\\+16B_{11}G_{12}^2I_0\end{array}}\right)\omega\end{array}\right)^{\frac{1}{2}}$$

$$r_{6g}=r_{4g}=-\frac{1}{2B_{11}\left(G_{11}h^2-2G_{12}\right)}\left(\begin{array}{l}-2B_{11}\left(G_{11}h^2-2G_{12}\right)\times\\ \left(I_0B_{11}h^2\omega-\sqrt{\begin{array}{l}I_0B_{11}^2h^4\omega^2+4B_{11}G_{11}^2h^4I_0\\-16B_{11}G_{11}h^2I_0G_{12}\\+16B_{11}G_{12}^2I_0\end{array}}\right)\omega\end{array}\right)^{\frac{1}{2}}$$

Applying the various boundary conditions, the obtained equations in matrices form are given as:

Table3 Boundary conditions with shear and with warping effects

Boundary Condition	Associated Equation in matrix form
Simply Supported	$\begin{bmatrix}0&0&1&1&1&1\\0&0&r_{3g}^2&r_{4g}^2&r_{5g}^2&r_{6g}^2\\0&0&e^{r_{3g}L}&e^{r_{4g}L}&e^{r_{5g}L}&e^{r_{6g}L}\\0&0&r_{3g}^2e^{r_{3g}L}&r_{4g}^2e^{r_{4g}L}&r_{5g}^2e^{r_{5g}L}&r_{6g}^2e^{r_{6g}L}\\1&1&0&0&0&0\\r_{1g}e^{r_{2g}L}&r_{2g}e^{r_{2g}L}&0&0&0&0\end{bmatrix}\begin{Bmatrix}A_{0g1}\\A_{0g2}\\B_{0g1}\\B_{0g2}\\B_{0g3}\\B_{0g4}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}$
Clamped Free	$\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\\0&0&r_{3g}&r_{4g}&r_{5g}&r_{6g}\\r_{1g}e^{r_{1g}L}&r_{2g}e^{r_{2g}L}&0&0&0&0\\1&1&r_{3g}^2e^{r_{3g}L}&r_{4g}^2e^{r_{4g}L}&r_{5g}^2e^{r_{5g}L}&r_{6g}^2e^{r_{6g}L}\\0&0&r_{3g}^2e^{r_{3g}L}&r_{4g}^2e^{r_{4g}L}&r_{5g}^2e^{r_{5g}L}&r_{6g}^2e^{r_{6g}L}\end{bmatrix}\begin{Bmatrix}A_{0g1}\\A_{0g2}\\B_{0g1}\\B_{0g2}\\B_{0g3}\\B_{0g4}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}$
Clamped Clamped	$\begin{bmatrix}1&1&0&0&0&0\\0&0&1&1&1&1\\0&0&r_{3g}&r_{4g}&r_{5g}&r_{6g}\\e^{r_{1g}L}&e^{r_{2g}L}&0&0&0&0\\1&1&e^{r_{3g}L}&e^{r_{4g}L}&e^{r_{5g}L}&e^{r_{6g}L}\\0&0&r_{3g}e^{r_{3g}L}&r_{4g}e^{r_{4c}L}&r_{5g}e^{r_{5g}L}&r_{6g}e^{r_{6g}L}\end{bmatrix}\begin{Bmatrix}A_{0g1}\\A_{0g2}\\B_{0g1}\\B_{0g2}\\B_{0g3}\\B_{0g4}\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\\0\end{Bmatrix}$

The analytical natural frequencies are derived by the resolution of the matrix form defined in Table 3 applying Newton-Raphson scheme while the finite elements solutions are obtained by substituting $\left[Z\right]$ and $\left[Z_i\right]$ by their expressions in this case:

$$\left[Z\right]=\begin{bmatrix}1&0&-z&0&-\dfrac{z^3}{3h^2}\\0&1&0&1&0\end{bmatrix};\left[Z_i\right]=\begin{bmatrix}1&z&z^3&0&0\\0&0&0&0&0\\0&0&0&1&z^2\end{bmatrix}$$

with the same procedure presented in section (2.2.2).

3. Numerical Results and discussion

In this section, numerical applications of sandwich beam using the three displacements field forms presented in the previous sections and subjected to various boundary conditions will be illustrated. The dynamic behavior of the sandwich beam in terms of natural frequencies are analyzed and compared for various displacement field forms.

The sandwich beam is constituted by three layer (N_c=3) with two faces made of Aluminum and a homogeneous Honeycomb core materials (Fig.2). The mechanical and geometrical characteristics of the sandwich beam are shown in Table 4.

Table4 Characteristics of the sandwich beam: Aluminum/Honeycomb/Aluminum

Elastic faces	Young Modulus Poisson ratio Mass density Thickness	$E_f=70\times10^9$ N/m^2 $\upsilon_f=0.3$ $\rho_f=2700$ Kg/m^3 $e_f=0.75$ mm
Homogeneous Honeycomb core	Young Modulus Poisson ratio Mass density Thickness Shear Modulus	$E_c=130\times10^6$ N/m^2 $\upsilon_c=0.33$ $\rho_c=573$ Kg/m^3 $e_c=5$ mm $G_c=5600\times10^6$ N/m^2
Sandwich beam Dimensions	Length Width	L= 250 mm b=53 mm

Fig.2. Sandwich beam configuration Al/Honeycomb/Al

The obtained results in terms of natural frequency of vibration for the various studied boundary conditions are shown in (Tables 5, 6 and 7). Then, they are compared in terms of relative error, as presented respectively in (Tables 8, 9 and 10), in order to analyze the effect of the displacement field form on the vibration characteristics of the sandwich beam.

Table5. Comparison of natural frequencies of a Clamped-free sandwich beam

Mode N°	SCG (FEM) [Hz]	SCG (Analytical) [Hz]	ACSG [Hz]	ACG [Hz]
1	55.11	55.25	57.68	59.2
2	332.8	333.1	348.45	365.44
3	824.48	825.53	871.23	898.35
4	1138.1	1142.35	1398.43	1560.77
5	1339.5	1340.34	1444.28	1687.23
6	1765.4	1753.97	1820.18	2154.23
7	1803.9	1806.03	2200.23	2564.12
8	2246	2265.05	2600.65	3120.44
9	2645.23	2648.78	3170.35	3545.81
10	3635.18	3633.45	4101.23	5120.56

Table6. Comparison of natural frequencies of a Simply-supported sandwich beam

Mode N°	SCG (FEM) [Hz]	SCG (Analytical) [Hz]	ACSG [Hz]	ACG [Hz]
1	306.41	306.73	310.07	312.34
2	602.29	602.39	654.76	658.76
3	682.3	682.32	703.38	783.98
4	1115.5	1114.11	1250.04	1434.12
5	1177.1	1182.69	1276.87	1598.01
6	1411.5	1421.69	1587.12	1723.3
7	1529.4	1555.94	1733.23	2045.76
8	1780.2	1783.96	2340.77	2780.34
9	1787.9	1791.77	2567.34	4873.02
10	2056.5	2136.7	3000.2	5120.45

Table7. Comparison of natural frequencies of a Clamped-clamped sandwich beam

Mode N°	SCG (FEM) [Hz]	SCG (Analytical) [Hz]	ACSG [Hz]	ACG [Hz]
1	361.45	361.5	361.7	360.98
2	739.20	739.64	738.76	737.99
3	815.5	815.92	817.98	820.23
4	1185.9	1201.07	1250.77	1310.34
5	1494.6	1497.62	1522.16	1634.23
6	1632.6	1651.52	1688.12	1820.73
7	2029.4	2052.74	2200.34	2500.34
8	2080.3	2132.80	2320.12	2783.92
9	2452.7	2469.47	2470.34	3132.84
10	2524.8	2527.35	2745.98	4567.71

The relative error associated to natural frequencies $\left(\varepsilon_f\right)$ is evaluated using the following expression: $\varepsilon_f\left(\%\right)=100\times\left|f^{ref}-f^{cal}\right|/f^{ref}$

With: f^{cal} is the natural computed frequency and f^{ref} is the reference natural frequency.

Tables 8, 9 and 10 shows that the natural frequencies of the sandwich beam derived from (FEM) and those derived from analytical solutions, in the case where shear and warping parameters are not considered (SCG), are in good agreement. In fact, the relative error does not exceed 3% for the ten modes in the studied frequency band for all boundary conditions. This satisfactory between (FEM) and analytical natural frequencies leads to validate the proposed analytical solutions in this (SCG) case. Then, it is compared to others analytical solutions where shear and warping parameters are considered: (ACSG) or (ASC) cases.

As can be seen from Table 8, the introduction of only shear parameter induces a smaller increase in natural frequencies relative to (SCG) case while the addition of warping parameter leads to significant rise in natural frequencies which reaches 41% with 10th mode. This effect can be explained by the addition of high term $\frac{z^3}{3h^2}\frac{\partial w_0\left(x,t\right)}{\partial x}$ which indicates a cubic variation of the displacements through the thickness of the sandwich beam generating hence a parabolic distribution of shear deformations.

Table8. Relative errors between three displacements forms for Clamped-free sandwich beam

Mode N°	$\varepsilon_f^{(SCG)}(\%)$	$\varepsilon_f^{(ACSG)}(\%)$	$\varepsilon_f^{(ACG)}(\%)$
1	0.25	4.39	7.14
2	0.09	4.60	9.70
3	0.46	5.53	8.82
4	0.37	22.42	36.63
5	0.06	25.89	25.89
6	0.64	3.78	22.83
7	0.11	21.83	41.98
8	0.84	14.81	37.77
9	0.11	19.69	33.68
10	0.05	12.78	40.92

Table9. Relative errors between three displacements forms for Simply-supported sandwich beam

Mode N°	$\varepsilon_f^{(SCG)}(\%)$	$\varepsilon_f^{(ACSG)}(\%)$	$\varepsilon_f^{(ACG)}(\%)$
1	0.1	1.08	1.82
2	0.02	8.69	9.35
3	0.002	3.08	11.81
4	0.12	12.20	28.72
5	0.40	7.96	35.11
6	0.70	11.63	21.21
7	1.70	11.39	31.48
8	0.20	31.21	55.85
9	0.21	43.28	171.96
10	3.89	40.42	139.65

Table10. Relative errors between three displacements forms for Clamped-clamed sandwich beam

Mode N°	$\varepsilon_f^{(SCG)}(\%)$	$\varepsilon_f^{(ACSG)}(\%)$	$\varepsilon_f^{(ACG)}(\%)$
1	0.01	0.05	0.15
2	0.05	0.12	0.10
3	0.05	0.25	0.53
4	1.27	4.14	9.10
5	0.20	4.94	8.28
6	1.15	2.22	10.25
7	1.15	7.20	21.80
8	2.51	8.79	30.53
9	0.68	0.03	26.87
10	0.10	8.66	80.73

Furthermore, the relative error of the 9th mode reaches 171.96% in the case of simply supported sandwich beam (Table9). This leads to conclude that the warping phenomenon generate an amplification on natural frequencies which must be controlled in the design of industrial sandwich structures. This fact is affirmed with the Clamped-clamped sandwich beam in which the natural frequencies increase especially for the last vibration modes (Table 10). Thus, the enrichment of the displacement field form by shear and warping effects increases the natural frequencies for each studied boundary conditions. Consequently, the inclusion of shear and warping parameters increases significantly the natural frequencies of the sandwich beam which must be considered in the analysis for such structures. The more the form of displacements field is enriched by shear and warping parameters, the better is the prediction of dynamic behavior of sandwich beams.

Conclusions

1) Three displacements field forms based on the introduction of shear and warping effects have been developed for the purpose of studying the free vibration analysis of laminated sandwich beams.
2) Analytical and finite elements (FEM) solutions are established for various boundary conditions.
3) Natural frequencies are computed and compared without and with the inclusion of shear and warping parameters.
4) The comparison study shows that the introduction of these parameters increases the natural frequencies especially for the last three modes in the frequency band of interest.
5) This allows controlling the dynamic behavior of such structures with regard to the potential dynamic calculation of the complex mechanical structures.
6) The use of various displacement forms and various boundary conditions enables us a good estimation of free vibration for flexure problems of laminated sandwich beams.

Future work

An experimental investigation of a honeycomb sandwich beam will be presented in future work to more validate these numerical results.

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