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Research Article

Dynamic Response of the Flexible Links and Joints of a Robot

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Abstract

This paper mainly focuses on the theoretical development of the equations of motion of a robotic manipulator, involving both joint flexibility and structural flexibility. The objective of this work is to extract the dynamic equations and write a 'C' program to check the effect of joint flexibility on the system frequencies. The obtained outcomes of this paper showthat the fundamental frequency is not sensitive to hub inertia or payload inertia but are mainly affected by payload mass, while the second frequency is affected by payload inertia.

Keywords: *manipulator, dynamic, payload, hub inertia, joint stiffness, fundamental frequency.*

1. Introduction

Robots are widely used in automobile and aerospace industries where high precision along with good amount of production is involved. As rigid robots are very heavy and cannot be operated at high speeds, intensive research in Dynamic modeling and control of robots with flexible links are carried out on both the rigid body and elastic deformations. The Flexible robot system is governed by partial differential equation, which means that the System is of infinite dimensionality.

The robot arm is made up of series of links equipped with actuators. The last link is connected to an end effector (i.e. gripper). The system of links without the end effectors is referred to as manipulator. The flexibility of robotic manipulator is to be analyzed for better design. The flexibility of robotic beam has received wide spread attention in connection With practical interest in applications such as flexible arm, helicopter rotor blades, air Craft propellers turbine rotor blades and space crafts.

2. Definition of Robot

"A reprogrammable multifunctional manipulator designed to move material, parts, tools or other specialized devices through variable programmed motions for performing variety of tasks."

3. Laws of robotics

1. A robot may not injure a human being or, through inaction, allow a human being to come to harm.
2. A robot must obey the orders given to it by human beings, except where such orders would conflict with the First Law.
3. A robot must protect its own existence as long as such protection does not conflict with the First or Second Law.

4. Configurations of robotic manipulators

a) *Cartesian co-ordinate robot:* The Cartesian co-ordinate robot is one that consists of a Column and an arm. It is sometimes called an x-y-z robot, indicating the axes of motion. The x-axis is lateral motion, the y-axis is longitudinal motion, and the z-axis is vertical motion.
b) *Cylindrical co-ordinate robot:* The cylindrical co-ordinate robot is a variation of the Cartesian robot. This robot is a variation of the column is able to rotate.
c) *Polar Co-ordinate robot:* It consists of a rotary base, an elevation pivot, and a telescoping extend and retract boom axis. It works on the spherical co-ordinate system and offers greater flexibility.
d) *Joint arm robot:* This resembles a robotic arm. It usually stands on a base on which it can rotate while it can articulate at the shoulder joint which is just above the base.

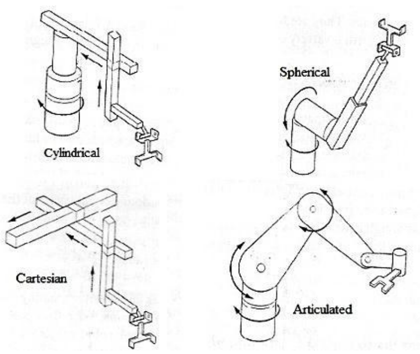

Fig1: Configurations of robot

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4. Robot manipulator motion

Six degrees of freedom are intended to emulate the versatility of movement possessed by Human arm. It consists of 3 arm & body motions and 3 wrist motions. Vertical traverse, radial traverse, rotational traverse are arm and body motions and swivel, bend and yaw are the wrist motions. All robots are not equipped with the ability to move in all six degrees.

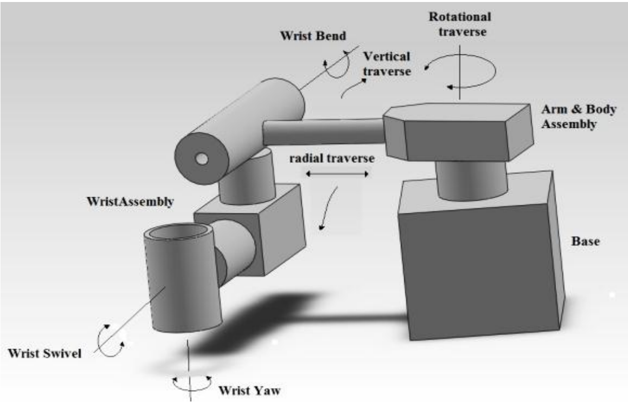

Fig2: manipulator motions

The major components of a robotic manipulator include the power conversion unit, the controller, sensory devices and manipulator.

Manipulator is integration of mechanical links connected by joints to form an open loop kinematic chain.

The controller will initiate & terminate the motion of components, store the data and permit the robot to be interfaced with the end users.

5. Problem formulation

To give the equations of motion for a robot manipulator, a uniformly distributed beam which is driven by a motor on one side through a flexible joint and a pay load on the other end is considered.

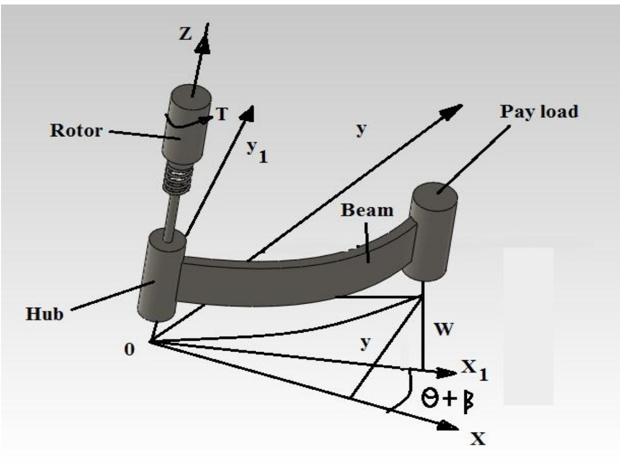

Fig 3: flexible beam with a flexible joint

The system has the following parameters:

L = Length of the link
E = Young's modulus of the link
I = Moment of inertia of the cross-section of the link
ρ= density (mass per unit length of link)
$\mathbf{m_{p=}}$ mass of pay load
$\mathbf{I_p}$=moment of inertia of the payload
$\mathbf{I_r}$ =moment of inertia of the rotor
K = stiffness of the joint
θ = the nominal position of the lonk with respect to the inertial frame XYZ
β = the angle of the hub with respect to rotor due to joint deflection.

The coordinate frame **XYZ** is the original inertial coordinate system with **Z** in vertical Direction, while $\mathbf{X_1Y_1Z_1}$ is the coordinate system attached to the hub, with $\mathbf{Z_1}$coincidingwith **Z**.

a) Assumptions

- Only the rigid motion and elastic deflection in the horizontal plane are considered.
- The elastic deflection is small.
- The radius of the joint hub and the pay load holding distance are ignored but their Moments of inertia are still taken into account.
- Bernoulli-euler beam assumptions are used (rotary inertia and shear are taken into account).
- The flexible joint is modeled as linear torsion spring.

Based on the above assumptions, the motion of the link is a superposition of the link's Gross motion and its transverse about its normal ride position. The position of any Point on the link is described by three quantities; **θ** represents the nominal position of the link with respect to the frame **XYZ; β** is the angle of the hub; and $\mathbf{W\ (X_1,t)}$ is the deflection of the link measured from the line $\mathbf{OX_1}$.

For small **θ** and **β**, the position of an arbitrary point **X** on the link is calculated as:

$$Y(X,t)=X[\theta(t)+\beta(t)]+W(X,t) \quad (1)$$

The kinetic energy of the system is expressed as

$$T =1/2I_r\theta^2+ ½\ I_h\ (\theta+\beta)\ 2+1/2\ {}_0\int^l \rho\ [X\ (\theta+\beta)+\delta w/\delta t]^2dx +1/2\ m_p[X(\theta+\beta)+\delta w/\delta t]_{x=l} +1/2\ i_p[(\theta+\beta)+\delta^2 w/\delta x\delta t]^2_{x=l} \quad (2)$$

The potential energy of the system is

$$V=1/2k\beta^2+1/2_0\int^l EI[\delta^2w/\delta x^2]^2dx \quad (3)$$

The only non-conservative force exerted on the system is the joint torque r. Its virtual Work is

$$\delta W=r\delta\theta \quad (4)$$

Where δθ is the virtual displacement of the rotor.

By applying hamilton's principle

$$\Delta_{t1}\int^{t2} (T-V+W)dt=0$$

A fourth order partial differential equation is obtained

$$EI(\delta^4 y/\delta x^4)+\rho(\delta^2 y/\delta t^2)=0 \quad (5)$$

With the boundary conditions

At X=0,

$Y_{x=0}=0$.… (6a)

$$(\delta y/\delta x)_{x=0}-(\theta+\beta)=0 \quad (6b)$$

At X=L

$$EI(\delta^3 y/\delta x^3)_{x=l}-m_p(\delta^2 y/\delta t^2)_{x=l}=0 \quad (6c)$$

$$EI(\delta^2 y/\delta x^2)_{x=l}+I_p(\delta^3 y/\delta x\delta t^2)_{x=l}=0 \quad (6d)$$

In additional, two dynamic equations can be established for rotor and the hub.

For the rotor:

$$T+K\beta=I_r\theta'' \quad (7)$$

For the hub

$$EI(\delta^2 y/\delta x^2)_{x=0}-K\beta=i_h(\theta''+\beta) \quad (8)$$

To derive the natural vibration modes, r is dropped from the equation (7), thus, the Equation of motion of the rotor is

$$\theta''=K/I_r(\beta) \quad (9)$$

differentiating the boundary conditions (6b) twice with respect to time and then substituting the result into equation (8), the following equation is obtained.

$$\beta=EI(\delta^2 y/\delta x^2)X_{=0}-I_h/K(\delta^3 y/\delta x\delta t^2)_{x=0} \quad (10)$$

b) Eigen Value Problem

Assume the harmonic motion of the link given by

$$Y(x,t)=\phi(x)\cos\omega t \quad (11)$$

Where ϕ (x) is the mode shape function of the link expressed in the coordinate system XYZ, and ω is the natural frequency. The motion of the flexible joint can be obtained From equation (10) as follows

$$B=1/K[EI\phi''(0)+I_h\omega^2\phi'(0)]\cos\omega t \quad (12)$$

Equation (12) is then substuted into equation (9) to get the motion of the rotor

$$\theta''=1/I_r[EI\phi''(o)+I_h\omega^2\phi'(0)]\cos\omega t \quad (13)$$

the solution to equation (13) is

$$\text{Ø}(t)=H\cos\omega t+c_1 t+c_2 \quad (14)$$

Where H is given by

$$H=-1/I_r[(EI/\omega^2)\phi''(0)+I_h\phi'(0)] \quad (15)$$

And C_1 and C_2 are integral constants.

For a flexible link, two types of natural modes are usually considered: the unconstrained and the constrained modes of vibration (20). The unconstrained modes are defined as the Modes of the link with all external influences removed. These modes correspond to the Motion of a flexible link in the free space (17).

The constrained modes are defined as those when rigid body is constrained or fixed in an inertial frame. The constrained modes are quite natural for a flexible link, rigid-joint system where the links are controlled by actuators. In this paper, the constrained modes are considered for the flexible-link, flexible-joint system. To derive the constrained modes, the rotor is fixed, which means that Ø=0

The characteristic equation of the problem can be derived as

$$B_1SCh+B_2CSh+B_3CCh+B_4SSh+B_5=0 \quad (16)$$

Where a shorthand notation S= sinλL, C =CosλL,

Sh=Sinhλl,andch=coshλl is usedλ is expressed as

$$\lambda^4=\rho\omega^2/(EI) \quad (17)$$

Bi in in the equation (16) has following forms

$$B_1=b_1+b_2-b_3+b_1b_2b_3 \quad (18a)$$

$$B_2=b_1-b_2+b_3-b_1b_3-1 \quad (18b)$$

$$B_3=b_1b_2-2b_1b_3-1 \quad (18c)$$

$$B_4=2b_2b_3-1 \quad (18d)$$

$$B_5=-b_1b_2-1 \quad (18e)$$

Where

$$B_1=1/3K_{ip}\gamma^3 \quad (18f)$$

$$B_2=k_{mp}\gamma \quad (18g)$$

$$B_3=(\gamma)/[1/3K_{ih}\gamma^4-K_k] \quad (18h)$$

K_k,K_{ih},K_{mp},and K_{ip} are the relative stiffness of the joint, relative moment of inertia of the hib, relative mass of the payload, and relative moment of inertia of the payload respectively. They are all non-dimensional parameters with respect to the beam, defined by

$$K_k=KL/EI, K_{ih}=I_h/I_h, K_{mp}=m_p/PL. K_{ip}=I_p/I_b \quad (19)$$

Where I_b is the moment of inertia of the beam given by

$I_b = (1/3)\rho L^3$. is the root of the characteristic equation (16) defined by

$$\gamma=\lambda L \quad (20)$$

which is also non-dimensional parameter.

Substuting [(18a) – (18h)] into (16), the characteristic equation can be expressed in the following form:

$$K_k - 1/3\,\gamma^4 K_{ih} = \frac{(3-K_{mp}K_{ip}\gamma^4)(S\,Ch - C\,Sh) + 2\,K_{ip}\gamma^3 C\,Ch + 6K_{mp}\gamma S Sh\gamma}{3(C\,Ch+1) - K_{ip}\gamma^3(S\,Ch + C\,Sh) - 3K_{mp}\gamma\,(S\,Ch - C\,Sh) - K_{mp}\,K_{ip}\gamma^4(C\,Ch-1)}$$

It can be shown that if the payload and hub are ignored, i.e.$K_{mp}=0$, $K_{ip}=0$, $K_{ih}=0$,

Equation (21) reduces to the characteristic equation of the classic pinned-free beem.

$$\mathrm{Tan}\,\gamma = \tanh\gamma$$

When $K_k=0$ (zero stiffness), and to the characteristic equation of the classic clamped-Free beam.

$$1+\cos\gamma\cosh\gamma=0$$

When $K_k=\infty$ (infinite stiffness).

The mode shape of the flexible –link, flexible-joint system is derived as

$$\Phi(x) = C[(\cos\lambda x - \cosh\lambda x) + \varepsilon\,(\sin\lambda x - \sinh\lambda x) + \zeta(\sin\lambda x + \sinh\lambda x)] \quad (22)$$

Where C is a constant, and ε and ζ are given by

$$\zeta = 3\gamma/(K_{ih}\gamma^4 - 3K_k) \quad (23a)$$

$$\varepsilon = -\frac{3(C+Ch)+\zeta[3(S-Sh)+\gamma^3 K_{ip}(C+Ch)] - \gamma^3 K_{ip} + (s+Sh)}{3(S+Sh)+\gamma^3 K_{ip}(C-Ch)}$$

The root γ reflects the natural frequency ω. Their relationship can be determined from equations (17) and (20) as follows:

$$\omega=\gamma^2/L^2\sqrt{EI/\rho} \quad (24)$$

6. Program

```
#include<math.h>
#include<conio.h>
#include<stdio.h>
Void main( )
{
Double i, x, y;
IntI,j, count;

Double klarr[4]={0.0,0.05,0.,0.5};
Double klarr[5]={0,1,20,100,99999};

Clrscr( );
For(1=0;1<4;1++)
{
For(j=0;j<5;j++)
{
For(i=0.0;i<11;i+=0.001)
{
X=klarr[1]*pow(1,4)/3.0;
X=kkarr[j]-x;
Y=i*(sin(i)*cosh(i)-cos(i)*sinh(i))/cos(i)*cosh(i)+);
If((y-x)<0.000000)
Printf("0/0d",i);

}
}
}
}
```

7. Mode Analysis

The influences of different parameters, including joint stiffness K_k, hub inertia K_{ih}, pay Load mass K_{mp} and load inertia Kip, are investigated. For convenience, rewriting the Characteristic equation as

$$K_k=1/3\,\gamma\,4K_{ih}+G \quad (25)$$

Where

$$G=\gamma^*\frac{(3-K_{mp}K_{ip}\gamma^4)(S\,Ch-CSh) - 2K_{ip}\,\gamma^3 CCh + 6K_{mp}\,\gamma SSh}{3(C\,Ch+1) - K_{ip}\,\gamma^3(s\,Ch+CSh) - 3K_{mp}\,\gamma\,(S\,Ch - C\,Sh) - K_{mp}K_{ip}\,\gamma^4(C\,Ch-1)} \quad (26)$$

8. Effect of Joint Stiffness K_k

The relative joint stiffness defined in equation is a non-dimensional parameter representing the relative stiffness of the joint with respect to that of the beam. Its physical Meaning can be explained by studying a flexible –joint, rigid-link system, as shown in fig. 2(a),if the angular displacement of the link caused by a torque M is α_1, the joint stiffness is $K=M/\alpha_1$

For a rigid joint, flexible-link system, the angular displacement $\alpha_2$of the link at the free end, under the action of the same torque M,is$\alpha_2=(ML)/(EI)$

By rearranging the above equation, the stiffness of the link, in the sense of the angular displacement at the end point, is obtained as

$$(EI)/L = M/\alpha_2$$

Thus the relative joint stiffness can be expressed by

$$K_k = (KL)/(EI) = \alpha_{2/}\,\alpha_1$$

Equation (27) indicates that relative joint stiffness K_k is equal to the ratio of angular displacement α_2 of the rigid-joint, flexible link system. When K_k=1 the flexible joint and the flexible link are said to have the equivalent stiffness. In case $\alpha_{1=}$ $\alpha_{2,}$ means that a torque applied to the flexible-Joint, rigid-link system will produce the same angular displacement at the end point of the link as the torque is applied to the system.

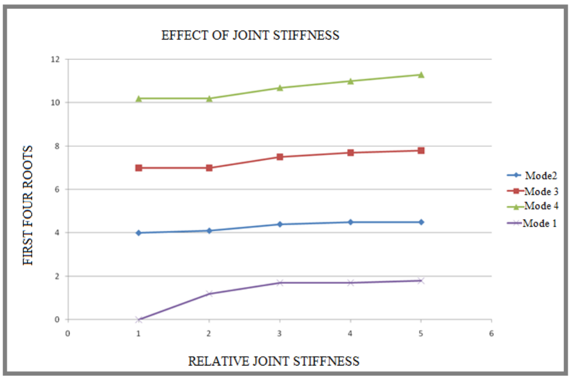

Fig 4:K_{ih}=0, k_{mp}=0, k_{ip}=0

Table1: Characteristic rootγ [K_{ih}=0, K_{mp}=0, K_{ip}=0,]

Mode	K_k=0	K_k=1	K_k=20	K_k=100	$K_k=\infty$
1	0	1.2479	1.7912	1.8568	1.8751
2	3.9266	4.0311	4.35127	4.6497	4.6941
3	7.0686	7.1341	7.5863	7.7827	7.8548
4	10.210	10.2566	10.6609	10.8976	10.9955

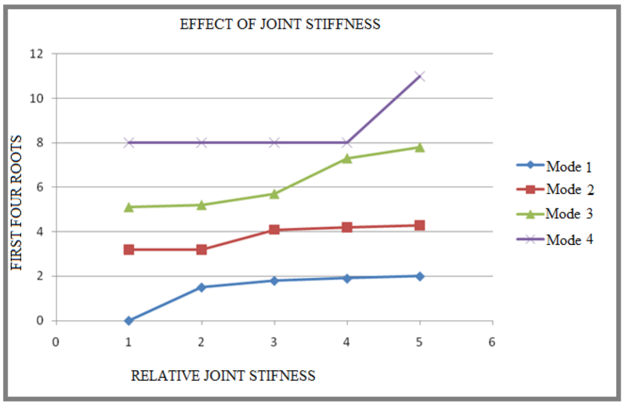

Fig 5:K_{ih}=0.1, k_{mp}=0, k_{ip}=0

Table2: Characteristic rootγ [K_{ih}=0.1, K_{mp}=0, K_{ip}=0,]

Mode	K_k=0	K_k=1	K_k=20	K_k=100	$K_k=\infty$
1	0	1.3412	1.8264	1.9312	1.9612
2	3.2146	3.2584	4.2241	4.3156	4.4112
3	5.1161	5.2163	5.6121	7.3412	7.8421
4	7.9112	7.9232	7.9432	8.0012	11.1324

Fig4 illustrates the effect of the relative joint stiffness K_k on the first four roots γ_i, Where K_k varies from 0.1 to 1000.four cases are studied for K_{ih} =0,0.05,0.1 and 0.5 Respectively, with zero payload (K_{mp} =0, K_{ip} =0). In case of K_{ih} =0, the system Consisting of a simple beam supported by a torsion spring is the simplest flexible-link, Flexible joint system. The value of γ_i for K_{ih} =0 and K_{ih} =0.1 are listed in table 1 and table 2 respectively. The roots for K_k=0 corresponding to the frequencies of the pinned-free beam, where the zero root implies the rigid body mode. The roots for $K_k=\infty$ corresponding to the frequencies of the clamped-free beam.

In order to study the sensitivity of Ɣ with respect to K_k, equation (25)is differentiated With respect to K_k, and the following equation is obtained.

$$\Delta y/\delta K_k=3/(4\gamma^3 K_{ih}+3\delta G/\delta\gamma) \quad (28)$$

Define the sensitivity index of γ with respect to $K_k$by(21)
S_k = (%change in γ)/ (%change in K_k) = $(\delta\gamma / \gamma)/(\delta K_k/K_k)$

(29)

In follows from (28) that

$$S_k=K_k/\gamma*[3/ (4\gamma^3 K_{ih}+3\delta G/\delta\gamma)] \quad (30)$$

The sensitivity index S_k for the above-mentioned four causes is shown in fig.6.

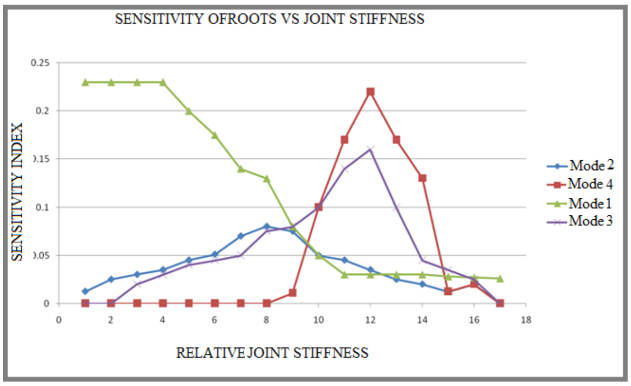

Fig 6:K_{ih}=0.05, k_{mp}=0, k_{ip}=0

Table3: Sensitivity Index S_k[K_{ih}=0.05, K_{mp}=0, K_{ip}=0,]

Mode	K_k=0	K_k=1	K_k=20	K_k=100	$K_k=\infty$
1	0.25	0.2019	0.042	0.0096	0
2	0	0.0225	0.0309	0.009	0
3	0	0.0085	0.0239	0.0085	0
4	0	0.0043	0.0186	0.008	0

The following conclusions can be drawn from the simulation results

1. Natural frequencies of the constructed system increase monotonically with the joint stiffness K_k and the simulated first four frequencies have the same tendency. This is in Contrast with the work by Xi and fenton on the unconstructed rotor beam system. In their work not only different frequencies have different tendencies, but also the same frequencies has different tendencies (namely, frequencies are not monotonic)

2. Natural frequencies approach constant values when K_k tends to infinity. The constant values are the frequencies of the clamped –free beam. Rewriting the equation (29) as

$y/\delta k_k=(y/K_k)*S_k$ (31)

3. The fundamental frequency is relatively sensitive to the joint stiffness K_k when K_k is small, but is insensitive when K_k is large. It should noted that the sensitivity index of $S^1=[\delta\omega/\omega] / [\delta K_k/K_k]$

Substuting quation (24) into the above equation yields $S^1=2*[\delta\gamma /\gamma] / [\delta K_k/K_k] = 2S_k$

4. Even as small flexibility (large stiffness) of the joint can significantly affect the system frequency. The frequency error caused ignoring the joint flexibility,
Error = $[1-[\{(\omega)K_k\}/\{(\omega)K_k=\infty\}]$

Using equation (24) the above equation can be expressed as
$\text{Error}=1-[(\gamma K_k)/(\gamma K_k=\infty)^2]$ (32)

Effect of Hub inertia K_{ih} :

Relative hub inertia K_{ih} defined by(19) ,is the ratio of the inertia of the hub to that of the beam. A large K_{ih} means relatively large hub inertia and relatively small beam inertia, and vice-versa. Differentiation of equation (25) with respect to K_{ih} yields
The following equation:

$$\frac{\delta\gamma}{\delta K_{ih}} = \frac{\gamma^4}{4\gamma 3K_{Ih} + 3\delta G/\delta\gamma}$$

similarly to (29), the sensitivity index of γ with respect to K_{ih} is defined as

$$S_{ih} = \frac{\delta\gamma/\gamma}{\delta K_{ih}/K_{ih}} = - \frac{\gamma^3 K_{ih}}{4\gamma^3 K_{ih} + 3\delta G/\delta\gamma} \quad (33)$$

fig 7 illustrates the effect of relative hub inertia of the first four roots γ_i, where K_{ih} varies from 0.01 to 100 . Four cases are studied for K_k = 0.1,1.0,20 and 100. The sensitivity index Sih for these four cases is shown in fig 6 .the values of γ and S_{ih} for case of K_k = 10,K_{mp} =0 and K_{ip} =0 are listed in tables respectively.

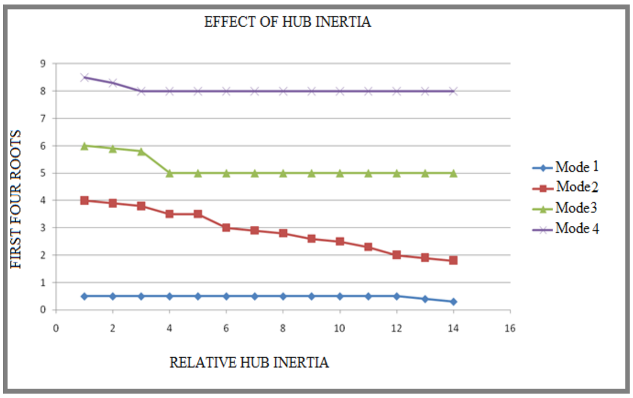

Fig 7:K_p=0.1, k_{mp}=0, k_{ip}=0

Table4: Characteristic rootγ [K_{ih}=10, K_{mp}=0, K_{ip}=0,]

Mode	K_k=0	K_k=0.01	K_k=0.1	K_k=1	K_k=∞
1	1.7227	1.7224	1.7190	1.6803	0
2	4.3995	4.3753	4.0411	2.5914	1.8751
3	7.4511	7.1059	5.2076	4.7253	4.6941
4	10.5218	9.0634	7.9244	7.8610	7.8548

The following conclusions are drawn from simulation results

1. Natural frequencies decrease with the hub inertia K_{ih} and approaches constantvalues with K_{ih} tends to infinity the constant values of the frequenciesof the clamped-free beam. K_{ih}goes to infinity fundamental frequency approaches zero, corresponding to rigidbody mode.

2. Fundamental frequency is not sensitive to hub inertia when $K_{ih}< 1$. In this case the natural frequencies are almost constant and the sensitivity index is small

8. Effects of payload mass k_{mp} and payload inertia k_{ih}:

Relative payload mass Kmp is the ratio of payload mass to the beam mass. Relative payload inertia Kip is the ratio of the payload inertia (with respect to its mass Center) to the beam inertia (with respect to the joint). Large K_{mp} and K_{ip} indicates relatively large payload, and vice versa. When a flexible robot is holding a payload, will the payload significantly affect the system frequencies? How will the frequencies change with different payload?

In order to answer this question, the effects of payload are simulated. The characteristics roots γ versus the payload mass K_{mp} , where K_{mp} varies from 0.1 to 100. Four cases are simulated for K_k =0.1, 1.0, 10 and with the hub inertia andpay load inertia constant (K_{ih}=0.1,Kip=0). The first three cases corresponded to the flexible-joint, flexible-link system with different joint stiffness, while the case corresponds to a rigid joint, flexible-link system. Figure8 illustrates the characteristic roots varies the payload inertia kip where K_{ip} varies from 0.01 to 10. In all simulated cases, K_k takes different values:0.1,1.0,10 and, and K_{ih} and K_{mp} remain constant (K_{ih}=0.1, K_{mp}=0.2).

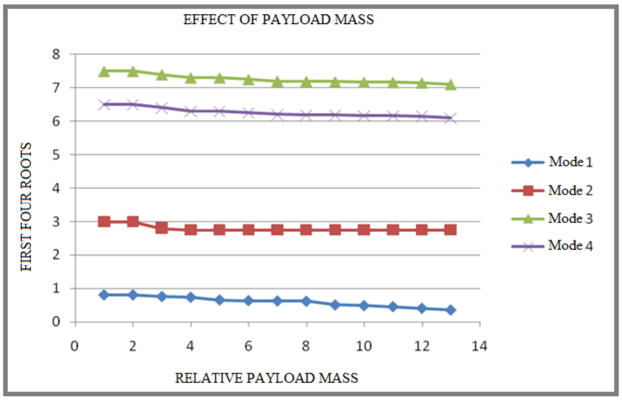

Fig 8: K_k=0.1, K_{mp}=0.1, K_{ip}=0

From the simulation results the following conclusions can be drawn

1. Payload mass affects the fundamental frequency, but does not significantly affect higher frequencies regardless of the flexible link, flexible-joint robots or the flexible link rigid joint robot.
2. The fundamental frequency insensitive to small payload inertia, but relatively sensitive to a large inertia; higher frequency are in sensitive to large payload inertia, but sensitive to a small inertia.
3. Among the first two frequencies, the fundamental frequency is mainly effected by the payload mass K_{mp}, while the second frequency is mainly effected by payload Inertia K_{ip}. This conclusion is useful in many engineering applications, while the flexible dynamics can be approximately represented by the first two modes.

This observation shows that for higher frequencies, payload inertia has a stronger effect then payload inertia does. For the fundamental frequency, payload mass has stronger effect then payload inertia does. This holds regardless of the joint Stiffness.

Conclusions

This paper presents the procedure for the extraction of the dynamic equations for a robot manipulator and also assists to check the effect of joint flexibility on the system frequencies. After studying the mode characteristics of the system, the following conclusions are made:

1. System frequencies are effected significantly even for a small joint flexibility.
2. The hub inertia or the payload inertia does not affect the fundamental frequency.
3. For a given flexible system, the fundamental frequency is mainly affected by the Payload mass, while the second frequency is mainly affected by the payload Inertia.

Using this approach, analysis for a multilink robot manipulator can be carried out. And controller design can be done using the observations of this paper.

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