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Research Article

A Study on Product Formulae for Mock Theta Functions of Different Order

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Abstract

Early in 1920, three months before his death, Ramanujan wrote his last letter to Hardy. In the course of it he said: I discovered very interesting functions in recent times which I call 'Mock θ- functions'. Unlike the 'False θ-functions', they enter into mathematics as wonderfully as the ordinary θ- functions. I am sending you with this letter some examples. The letter was accompanied by five foolscap passes. In the first three pages Ramanujan explained what he meant by a 'Mock θ- function'. Hardy's comment about Mock θ- function is; A Mock θ- function is a function defined by a q-series, convergent when |q|<1. We can calculate asymptotic formulae for it, when q tends to a rational point $e^{2\pi ir/s}$ of the unit circle of the same degree of precision these furnished for the ordinary θ- function by the theory of linear transformation.

Keywords: *Hypergeometric functions, Mock Theta function*

1. Introduction and Main results

The three pages of explanation are as follows; If we consider a θ - function in the transformed Eulerian form, e.g.

(A)

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2\left(1-q^2\right)^2}+\frac{q^9}{(1-q)^2\left(1-q^2\right)^2\left(1-q^3\right)^2}+\ldots$$

(B)

$$1+\frac{q}{(1-q)}+\frac{q^4}{(1-q)\left(1-q^2\right)}+\frac{q^9}{(1-q)\left(1-q^2\right)\left(1-q^3\right)}+\ldots$$

and determine the nature of singularities at the points

$$q=1, q^2=1, q^3=1, q^4=1, q^5=1, \ldots.$$

We know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when

$q=e^{-t}$ and $t\to 0$,

(A)

$$\sqrt{\frac{t}{2\pi}}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+0(1)t$$

(B)

$$\sqrt{\frac{2}{5-\sqrt{5}}}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+0(1)$$

and similar results at other singularities. If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above, but in a majority of cases they never close as above. For instance, when $q=e^{-t}$ and t = 0,

(C)

$$1+\frac{q}{(1-q)^2}+\frac{q^3}{(1-q)^2\left(1-q^2\right)^2}+\frac{q^5}{(1-q)^2\left(1-q^2\right)^2\left(1-q^3\right)^2}+\ldots$$

$$=\sqrt{\frac{t}{2\pi\sqrt{5}}}\exp\left[\frac{\pi^2}{5t}+a_1t+a_2t^2+\ldots+a_kt^k\right],$$

Where

$$a_1=\frac{1}{8\sqrt{5}}$$

and so on. The function (C) is a simple example of a function behaving in an unclosed form at singularities.

The Eulerian form of a function apparently refers to the character of the denominator of the terms in the series. The phrase is probably suggested by Euler's formula,

$$\sum_{n=0}^{\infty}\frac{q^{n(n-1)/2}z^n}{(q;q)_n}=(-z;q)_\infty,$$

$$\sum_{n=0}^{\infty}\frac{z^n}{[q;q]_n}=\frac{1}{[z;q]_\infty}.$$

(A) is immediately derivable from Hein's formula for basic hypergeometric series,

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$$ {}_2\Phi_1\left[\begin{matrix}a,b;c/ab\\ c\end{matrix}\right]=\frac{[c/a,c/b;q]_\infty}{[c,c/ab;q]_\infty} $$

By taking $a,b\to\infty$ and $c\to q$. (A) is thus the partition function $\frac{1}{[q;q]_\infty}$.

The function (B) is G (q) where

$$ G(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{[q;q]_n}=\frac{1}{\left[q;q^5\right]_\infty\left[q^4;q^5\right]_\infty}. $$

Ramanujan further stated that Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true ?

Suppose that there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). the question is: Is the function can be taken as the sum of two functions one of which is an ordinary theta function and the other is a function which is 0(1) at all the parts $e^{2\pi ir/s}$? the answer is it is not necessary so. When it is not so, I call the function a Mock theta function.

The last two pages of Ramanujan's notes consists of list of definitions of four sets of Mock theta functions with statements of relations connecting members of each of the first three sets; for obvious reasons the functions in the sets are described as being of orders 3, 5, 5, and 7 respectively. Ramanujan has discovered four functions of order three. He seems to have overlooked the existence of the set of functions *ω(q), ν (q)* and *ρ(q)* a which were discovered by G.N. Watson. The complete set of mock theta functions of order three are;

$$ f(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n^2}\qquad \Phi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[-q^2;q^2\right]_n} $$

$$ \Psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[q;q^2\right]_n}\qquad \chi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[-\omega q,-\omega^2 q;q\right]_n} $$

$$ \omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left[q;q^2\right]_{n+1}^2}\qquad \rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left[\omega q,\omega^2 q;q^2\right]_{n+1}} $$

$$ \nu(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\left(-q;q^2\right)_{n+1}} $$

(2) These several functions are connected by the following relations

$$ 2\Phi(-q)-f(q)=f(q)+4\Psi(-q)=\frac{\theta_4(q)}{(-q;q)_\infty} $$

$$ 4\chi(q)-f(q)=\frac{3\theta_4^2\left(q^3\right)}{(q;q)_\infty} $$

$$ 2\rho(q)+\omega(q)=\frac{3}{4}\frac{\theta_2^2\left(q^{3/2}\right)}{q^{3/4}\left(q^2;q^2\right)_\infty} $$

$$ \nu(\pm q)\pm q\omega\left(q^2\right)=\frac{\theta_2(q)}{2q^{1/4}\left(-q^2;q^2\right)_\infty} $$

$$ f\left(q^8\right)\pm 2q\omega(\pm q)\pm 2q^3\omega\left(-q^4\right)=\frac{\theta_3(\pm q)\theta_3^2\left(q^2\right)}{\left(q^4;q^4\right)_\infty^2} $$

2. Transformation formulae for Mock theta functions order three

In this section we need the following identity in order to establish our transformation of mock theta function order three.

$$ \sum_{n=0}^{\infty}\frac{[aq/bc;q]_n\Omega_n x^n}{[q,aq/b,aq/c;q]_n}=\sum_{n,k=0}^{\infty}\frac{[a,b,c;q]_k\left(1-aq^{2k}\right)(-axq/bc)^k q^{\binom{k}{2}}\Omega_{n+k}x^n}{[q,aq/b,aq/c;q]_k(1-a)[aq;q]_{2k+n}[q;q]_n}, \tag{1} $$

where Ω_n is an arbitrary sequence and provided all the series involved are convergent. In order to prove (1), we consider the following known results [Gasper G. & Rahman M.[2];App. II; (11.21)].

$$ \sum_{k=0}^{\infty}\frac{\left[a,b,c,q^{-n};q\right]_k\left(1-aq^{2k}\right)\left(aq^{n+1}/bc\right)^k}{\left[q,aq/b,aq/c,aq^{n+1};q\right]_k(1-a)}=\frac{[aq,aq/bc;q]_n}{[aq/b,aq/c;q]_n}. \tag{2} $$

Now, multiplying both sides of (2) by

$$ \frac{\Omega_n x^n}{[q,aq;q]_n} $$

and summing over n from 0 to infinity, we get

$$ \sum_{k=0}^{\infty}\frac{(aq/bc;q)_n\Omega_n x^n}{(q,aq/b,aq/c;q)_n}=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{\left[a,b,c,q^{-n};q\right]_k\left(1-aq^{2k}\right)\left(aq^{n+1}/bc\right)^k\Omega_n x^n}{\left[q,aq/b,aq/c,aq^{n+1};q\right]_k(1-a)[q,aq;q]_n}. \tag{3} $$

Now, applying the well-known identity,

$$ \sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k), $$

on the right of (3), we get (1), after some simplifications.
Assuming

$\Omega_n=[\alpha,\beta;q]_n$ and $x=aq/\alpha\beta$ in (1) we get

$$ \sum_{n=0}^{\infty}\frac{[\alpha,\beta,aq/bc;q]_n(aq/\alpha\beta)^n}{[q,aq/b,aq/c;q]_n} $$

$$ =\sum_{k=0}^{\infty}\frac{[a,b,c,\alpha,\beta;q]_k q^{k(k-1)/2}\left(1-aq^{2k}\right)\left(-a^2q/\alpha\beta bc\right)^k}{[q,aq/b,aq/c;q]_k[aq;q]_{2k}(1-a)}\,{}_2\Phi_1\left[\begin{matrix}\alpha q^k,\beta q^k;q;aq/\alpha\beta\\ aq^{2k+1}\end{matrix}\right]. $$

Summing the inner series on the right hand side of (4) with help of [Gasper, G. & Rahman, M. [2]; App. III (8)] we get

$$ \sum_{n=0}^{\infty}\frac{[\alpha,\beta,aq/bc;q]_n(aq/\alpha\beta)^n}{[q,aq/b,aq/c;q]_n} $$

$$=\frac{[aq/\alpha,aq/\beta;q]_\infty}{[aq,aq/\alpha\beta]_\infty}\sum_{k=0}^{\infty}\frac{[a,b,c,\alpha,\beta;q]_k\, q^{k(k-1)/2}\left(1-aq^{2k}\right)\left(-a^2q^2/\alpha\beta bc\right)^k}{[q,aq/b,aq/c;q]_k\,[aq/\alpha,aq/\beta;q]_k\,(1-a)} \tag{5}$$

Further, setting $\alpha,\beta\to\infty, b=c=-1$ and then $a\to 1$ in (5) we get

$$[q;q]_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n^2}=[q;q]_\infty f(q)=1+4\sum_{k=1}^{\infty}\frac{(-)^k q^{k(3k+1)/2}}{\left(1+q^k\right)}. \tag{6}$$

Next, letting $\alpha,\beta\to\infty, b=-c=iq$ and then taking $a\to 1$ in (5) we get

$$[q;q]_\infty\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[-q^2;q^2\right]_n}=[q;q]_\infty\Phi(q)=1+2\sum_{k=1}^{\infty}\frac{(-)^k\left(1+q^k\right)q^{k(3k+1)/2}}{\left(1+q^{2k}\right)}. \tag{7}$$

Replacing q by q^2 in (5) and then taking $\alpha=q^2, b,\beta\to\infty$ and $a=cq$ we get,

$$\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{[q;q]_{n+1}}=q\sum_{k=0}^{\infty}\frac{\left[-q^2;q^2\right]_k\left(1+q^{4k+3}\right)q^{3k(k+1)}}{\left[q;q^2\right]_{k+1}}$$

which yields

$$\Psi(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[q;q^2\right]_n}=q\sum_{k=0}^{\infty}\frac{\left[-q^2;q^2\right]_k\left(1+q^{4k+3}\right)q^{3k(k+1)}}{\left[q;q^2\right]_{k+1}} \tag{8}$$

Again, taking $\alpha,\beta\to\infty$, $b=-a/\omega, c=-a/\omega^2$ and then $a\to 1$ in (5) we get

$$[q;q]_\infty=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left(1-q+q^2\right)\ldots\left(1-q^n+q^{2n}\right)}=[q;q]_\infty\chi(q)$$

$$=1+\sum_{k=1}^{\infty}\frac{(-)^k\left(1+q^k\right)q^{k(3k+1)/2}}{\left(1-q^k+q^{2k}\right)}, \tag{9}$$

where

$\omega=e^{2\pi i/3}$

Further, assuming $\alpha,\beta\to\infty$, $a=bc$ and then $b=iq^{1/2}, c=-iq^{1/2}$ in (5) we get,

$$[q;q]_\infty\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{\left[-q;q^2\right]_{n+1}}=[q;q]_\infty\nu(q)=\sum_{k=0}^{\infty}\frac{(-)^k\left(1+q^{2k+1}\right)q^{3k(k+1)/2}}{\left(1+q^{2k+1}\right)}. \tag{10}$$

Next, letting $\alpha,\beta\to\infty, a=bc$ and then replacing q by q^2 in (5) we get,

$$\sum_{n=0}^{\infty}\frac{q^{2n^2}(bc)^n}{\left[bq^2,cq^2;q^2\right]_n}=\frac{1}{\left[bcq^2;q^2\right]_\infty}\times$$

$$\times\sum_{k=0}^{\infty}\frac{\left[bc,b,c;q^2\right]_k(-)^k(bc)^k\left(1-bcq^{4k}\right)q^{k(3k+1)}}{\left[q^2,bq^2,cq^2,q^2;q^2\right]_k(1-bc)}. \tag{11}$$

Now, taking $b=\omega q$, $c=\omega^2 q$ in (11) we get,

$$\rho(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left(1+q+q^2\right)\ldots\left(1+q^{2n+1}+q^{4n+2}\right)}$$

$$=\frac{1}{\left[q^2;q^2\right]_\infty}\sum_{k=0}^{\infty}\frac{(-)^k\left(1-q^{4k+2}\right)q^{3k(k+1)}}{\left(1+q^{2k+1}+q^{4k+2}\right)}. \tag{12}$$

Lastly, letting $\alpha,\beta\to\infty$ and then replacing q by q^2 and finally taking b = c = q and $a=q^2$ in (5) we get,

$$\omega(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left[q;q^2\right]_{n+1}^2}=\frac{1}{\left[q^2;q^2\right]_\infty}\sum_{k=0}^{\infty}\frac{(-)^k\left(1-q^{2k+1}\right)q^{3k(k+1)}}{\left(1-q^{2k+1}\right)}. \tag{13}$$

All these representations are single series representations.

3. Mock theta functions of order five

Ten mock theta functions of order five as,

$$f_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{[-q;q]_n}$$

$$f_1(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}}{[-q;q]_n}$$

$$\Phi_0(q)=\sum_{n=0}^{\infty}q^{n^2}\left[-q;q^2\right]_n$$

$$\Phi_1(q)=\sum_{n=0}^{\infty}q^{(n+1)^2}\left[-q;q^2\right]_n$$

$$\Psi_0(q)=\sum_{n=0}^{\infty}[-q;q]_n\, q^{(n+1)(n+2)/2}$$

$$\Psi_1(q)=\sum_{n=0}^{\infty}[-q;q]_n\, q^{n(n+1)/2}$$

$$F_0(q)=\sum_{n=0}^{\infty}\frac{q^{2n^2}}{\left[q;q^2\right]_n}$$

$$F_1(q)=\sum_{n=0}^{\infty}\frac{q^{2n(n+1)}}{\left[q;q^2\right]_{n+1}}$$

$$\chi_0(q)=\sum_{n=0}^{\infty}\frac{q^n}{\left[q^{n+1};q\right]_n}$$

$$\chi_1(q)=\sum_{n=0}^{\infty}\frac{q^n}{\left[q^{n+1};q\right]_{n+1}}$$

Seventh order mock theta functions

Perhaps the most mysterious of all the mock theta functions are the following seventh order functions,

$$\zeta_0(q)=\sum_{n=0}^{\infty}\frac{q^{n^2}}{\left[q^{n+1};q\right]_n}=\sum_{n=0}^{\infty}\frac{q^{n^2}[q;q]_n}{[q;q]_{2n}}$$

$$\zeta_1(q)=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}}{\left[q^{n+1};q\right]_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{(n+1)^2}[q;q]_n}{[q;q]_{2n+1}}$$

$$\zeta_2(q)=\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{\left[q^{n+1};q\right]_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{n(n+1)}[q;q]_n}{[q;q]_{2n+1}}$$

Ramanujan has asserted that all the three functions are unrelated. Watson, agreeing with Ramanujan's comment, did not study those functions.

Mock theta functions in the lost notebook

Andrews and Hickerson noted seven more q-series found in the 'lost' notebook and designated them as mock theta functions of order six. Seven functions are defined as

$$\Phi_L(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}\left[q;q^2\right]_n}{[-q;q]_{2n}}$$

$$\Psi_L(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{(n+1)^2}\left[q;q^2\right]_n}{[-q;q]_{2n+1}}$$

$$\rho_L(q)=\sum_{n=0}^{\infty}\frac{[-q;q]_n q^{n(n+1)/2}}{\left[q;q^2\right]_{n+1}}$$

$$\sigma_L(q)=\sum_{n=0}^{\infty}\frac{[-q;q]_n q^{n(n+1)(n+2)/2}}{\left[q;q^2\right]_{n+1}}$$

$$\lambda_L(q)=\sum_{n=0}^{\infty}(-)^n q^n\frac{\left[q;q^2\right]_n}{[-q;q]_n}$$

$$\mu_L(q)=\sum_{n=0}^{\infty}(-)^n\frac{\left[q;q^2\right]_n}{[-q;q]_n}$$

The series for μ *(q)* is a divergent series. The sequence of even partial sums converges, as does the sequence of odd partial sums. We take μ *(q)* as the average of these two values.

Mock theta functions of order ten

Recently, Y. Choi in 1999 published a long paper in which he citied four more q- functions found on page 9 of the 'lost' notebook and designated then as mock theta functions of order ten. These functions are,

$$\Phi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{\left[q;q^2\right]_{n+1}}$$

$$\Psi_{LC}(q)=\sum_{n=0}^{\infty}\frac{q^{(n+2)(n+1)/2}}{\left[q;q^2\right]_{n+1}}$$

$$\chi_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{n^2}}{\left[-q;q^2\right]_{2n}}$$

$$\chi'_{LC}(q)=\sum_{n=0}^{\infty}\frac{(-)^n q^{(n+1)^2}}{\left[-q;q^2\right]_{2n+1}}$$

Ramanujan's last gift to the world of mathematics is his class of Mock-theta functions which he classified into three categories, i.e., the mock theta functions of order 3,5 and 7. Watson added three more to the list of mock-theta functions of order three. There are two groups of mock-theta functions of order 5 about which he mentioned that these are connected among themselves but not to any one from order 3 or 7. He also mentioned that the three mock-theta functions of order 7 are neither connected among themselves nor to anyone else from any other group.

Later on, Andrews and Hickerson Gorden and Mc Intosh and Choi came across certain q-series in the 'Lost' Notebook and termed them as mock-theta functions of order 6,8 and 10 respectively. None of them, including Ramanujan provided any justification for their categorization into various classes.

No clue regarding the categorization of mock-theta functions has surfaced as yet. Agarwal showed that the mock-theta functions of order 3,5 and 7 are limiting cases of $2\Phi'^{s}1$, $3\Phi'^{s}2$, $4\Phi'^{s}3$ and respectively. This looks quite reasonable to classify them so, but the trouble starts when we deal with the mock-theta functions of order 6 and 10, all of which having one are the limiting cases $3\Phi'^{s}2$. We have not yet tried such representations for the mock-theta functions of order 8.

There are various representations of mock-theta functions given by Denis, Singh and Ahmad Ali and Denis, Singh and Singh and Fine, Watson and others. Except for the mock-theta functions of order 3, which have single series representations all the others, have double series representations that are not unique.

Ramanujan opined that the mock-theta function of various orders are not related to each other except the two groups of order 5. Perhaps what he meant is that these are not linearly related to each other. We (Denis et. al.) have dealt with this aspect at length.

4. Main Results (Product formulae involving mock theta functions)

Here our attempt will be to show that there exists relationship between any two arbitrary mock-theta functions belonging to the same order or different orders. However, we shall confine to the mock-theta functions of order 3 5 and 7 for obvious reason which does not in the last, excludes the relationship involving the recently termed mock-theta functions of order 6, 8 and 10.

We shall need the following known identities,

$$\sum_{m=0}^{n}\delta_m\sum_{r=0}^{m}\alpha_r=\sum_{r=0}^{n}\alpha_r\sum_{m=0}^{n}\delta_m-\sum_{r=0}^{n-1}\alpha_{r+1}\sum_{m=0}^{r}\delta_m \tag{14}$$

and

$$\sum_{k=0}^{n} a_k \sum_{j=0}^{n-k} A_j = \sum_{k=0}^{n} A_k \sum_{j=0}^{n-k} a_j, \tag{15}$$

where α_r , δ_r, a_r and $A_r$are arbitrary sequences.

Relationship between various mock-theta functions

Here we shall attempt to establish the relationship between mock-theta functions. Our attempt shall not be establish all the relations, but shall confine only to illustrating the method.

(i) If we take
$\alpha_r = \frac{q^{r^2}}{[-q;q]_r^2}$ and $\delta_m = \frac{p^{m^2}}{[-p;p]_m}$ in (15), with $|q|,|p|<1$, we get,

$$f_n(q)\, f_{0,n}(p) = \sum_{m=0}^{n} \frac{p^{m^2}}{[-p;p]_m} f_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}^2} f_{0,r}(q) \tag{16}$$

If $n\to\infty$ in (16), we get

$$f(q)\, f_0(p) = \sum_{m=0}^{\infty} \frac{p^{m^2}}{[-p;p]_m} f_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}^2} f_{0,r}(q) \tag{17}$$

The above relationship involves a mock-theta function of order 3 with base q and the other one, of the order 5, with base p.

(ii) Next, If we take
$\alpha_r = \frac{q^{r^2}}{[-q;q]_r}$ and $\delta_m = \frac{p^{2m(m+1)}}{\left[p;p^2\right]_{m+1}}$ in (16), we get,

$$f_{0,n}(q) F_{1,n}(p) = \sum_{r=0}^{n} \frac{p^{2r(r+1)}}{\left[p;p^2\right]_{r+1}} f_{0,r}(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}} F_{1,r}(p) \tag{18}$$

If $n\to\infty$ in (18), we get

$$f_0(q) F_1(p) = \sum_{r=0}^{\infty} \frac{p^{2r(r+1)}}{\left[p;p^2\right]_{r+1}} f_{0,r}(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}} F_{1,r}(p) \tag{19}$$

which is a relationship between two mock-theta functions of order 5, one from each group. Here to the base of one is p and that of the other q.

(iii) Again If we take

$\alpha_r = \frac{q^{r^2}}{[-q;q]_r^2}$ and $\delta_m = \frac{p^{m^2}}{\left[p^{m+1};p\right]_m}$ in (20), we get,

$$f_n(q)\Im_{0,n}(p) = \sum_{m=0}^{n} \frac{p^{m^2}}{\left[p^{m+1};p\right]_m} f_m(q) + \sum_{r=0}^{n-1} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}^2} \Im_{0,r}(p) \tag{21}$$

which, for $n\to\infty$ yields,

$$f(q)\Im_0(p) = \sum_{m=0}^{\infty} \frac{p^{m^2}}{\left[p^{m+1};p\right]_m} f_m(q) + \sum_{r=0}^{\infty} \frac{q^{(r+1)^2}}{[-q;q]_{r+1}^2} \Im_{0,r}(p) \tag{22}$$

Hence one mock-theta belongs to order three and the other one to order 7, both on different bases.

(iv) Further, if we set
$\alpha_r = \frac{q^{(r+1)^2}}{\left[q^{r+1};q\right]_{r+1}}$ and $\delta_m = \frac{p^{m(m+1)}}{\left[p^{m+1};p\right]_{m+1}}$ in (15), we get the following relationship between two mock-theta function of order 7,

$$\Im_{1,n}(q)\Im_{2,n}(p) = \sum_{m=0}^{n} \frac{p^{m(m+1)}}{\left[p^{m+1};p\right]_{m+1}} \Im_{1,n}(q) + \sum_{r=0}^{n-1} \frac{q^{(r+2)^2}}{\left[q^{r+2};q\right]_{r+2}} \Im_{2,r}(p) \tag{23}$$

If $n\to\infty$ in (23), we get

$$\Im_1(q)\Im_2(p) = \sum_{m=0}^{\infty} \frac{p^{m(m+1)}}{\left[p^{m+1};p\right]_{m+1}} \Im_{1,n}(q) + \sum_{r=0}^{\infty} \frac{q^{(r+2)^2}}{\left[q^{r+2};q\right]_{r+2}} \Im_{2,r}(p) \tag{24}$$

Lastly, we illustrate how (16) can be used to establish relationship between two arbitrary mock-theta functions.

If we take
$A_k = \frac{q^{2k(k+1)}}{\left[q;q^2\right]_{k+1}^2}$ and $a_k = \frac{p^{k(k+1)}}{[-p;p]_k}$ in (17), we get,

$$\sum_{k=0}^{n} \frac{p^{k(k+1)}}{[-p;p]_k} \omega_{n-k}(q) = \sum_{k=0}^{n} \frac{q^{2k(k+1)}}{\left[q;q^2\right]_{k+1}} f_{1,n-k}(p). \tag{25}$$

It is evident that such relationship will exist between any two arbitrary mock-theta functions.

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