Generalized Gaussian Quadrature Rules over an arbitrary cube in Euclidean Three-Dimensional Space

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Abstract

This paper presents a Generalized Gaussian quadrature method for the evaluation of volume integral

\[ I = \iiint f(x, y, z) \, dx \, dy \, dz, \]

where \( f(x, y, z) \) is arbitrary function and \( \Omega \) refers to the volume of cube \( \{ (x, y, z) : 0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1 \} \), to evaluate the typical volume integrals over the cubic region governed by the proposed method.

Keywords: Finite element method, Generalized Gaussian quadrature, cubic region.

1. Introduction

The finite element method has been used as a powerful tool to obtain approximate solution for mathematical problems. Such as to calculate stiffness matrix, to solve integral equation, volume, center of mass, moment of inertia and other geometric properties of rigid homogeneous solids frequently arise in large number of engineering applications in CAD/CAE/CAM, applications in geometric modeling as well as in robotics and similar problems in other areas of engineering which are very difficult to analyze using analytical technique. This problem can be solved using the finite element method.

In this paper, we attempt to extend our approach to derive new quadrature formula over cubic region. Triple integration of any arbitrary function over cubic region are calculated numerically by using generalized Gaussian quadrature rule. The integration points have to be increased in order to improve the integration accuracy. The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries required for understanding the derivation. In Section 3 we calculate the Generalized Gaussian quadrature nodes and weights for cube. We also plotted the Gaussian points for \( N = 5, 10, 15, 20 \). Section 4 we compare the numerical results with some illustrative examples.

2. Mathematical preliminaries

Let us consider the integral of the form

\[ \int_{a}^{b} q(x) \, \Phi(x) \, dx = \sum_{k=1}^{n} W_{k}^{(n)} \, \Phi(x_{k}) \]  

(1)

with \( x_{k} \in [a, b] \) and \( W_{k}^{(n)} \in R \) for all \( i = 1, 2, 3, \ldots, n \). The points \( x_{k} \) and coefficients \( W_{k}^{(n)} \) are referred to as nodes and weights coefficients.

2.1 Generalized Gaussian quadratures

In this paper, we attempt to extend our approach to derive new quadrature formula over cubic region. Triple integration of any arbitrary function over cubic region are calculated numerically by using generalized Gaussian quadrature rule. The integration points have to be increased in order to improve the integration accuracy. The remainder of this paper is organized as follows. Section 2 presents the mathematical preliminaries required for understanding the derivation. In Section 3 we calculate the Generalized Gaussian quadrature nodes and weights for cube. We also plotted the Gaussian points for \( N = 5, 10, 15, 20 \). Section 4 we compare the numerical results with some illustrative examples.

Let the Numerical integration of an arbitrary function \( f(x, y, z) \) over a cube is given by

\[ I = \iiint_{c} f(x, y, z) \, dx \, dy \, dz \]  

(2)

The integral of the equation (2) can be transform to square region \( \{ (\xi, \eta, \zeta) : 0 \leq \xi \leq 1, 0 \leq \eta \leq 1, 0 \leq \zeta \leq 1 \} \), mathematical transformation is

\[ x = \xi, \quad y = \eta, \quad z = \zeta \]  

(3)

\[ I = \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} f(x, y, z) \, dx \, dy \, dz \]
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\[ \int_0^1 \int_0^1 \int_0^1 f(x(\xi, \eta, \zeta), y(\xi, \eta, \zeta), z(\xi, \eta, \zeta)) \, dx \, dy \, dz \]

Where \( J = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{vmatrix} = 1 \)

eqn.(3) we can write as

\[ I = \int_0^1 \int_0^1 \int_0^1 f(\xi, \eta, \zeta) \, d\xi \, d\eta \, d\zeta \]

\[ = \sum_{i=1}^{n_x} \sum_{j=1}^{n_y} \sum_{k=1}^{n_z} w_i w_j w_k f(\xi_i, \eta_j, \zeta_k) \quad (4) \]

Where \( \xi_i, \eta_j, \zeta_k \) are Gaussian points and \( w_i, w_j, w_k \) are corresponding weights. We can rewrite eqn. (4) as

\[ I = \sum_{i=1}^{N} C_i f(x_i, y_i, z_i) \quad (5) \]

Where \( N = \alpha \beta \) is the total number of integration points

\[ C_i = w_i^{(\alpha)} w_j^{(\beta)} \quad (5a) \]
\[ x_i = \xi_i^{(\alpha)} \quad (5b) \]
\[ y_i = \eta_j^{(\beta)} \quad (5c) \]
\[ z_i = \zeta_i^{(a)} \quad (5d) \]

Where \( \xi_i^{(\alpha)} \) is the set of one – dimensional Gaussian points, \( \xi_i^{(\beta)} \) and \( \eta_j^{(\beta)} \) are set of two – dimensional Gaussian points.

we find out new Gaussian points \( x_i, y_i, z_i \) and weights coefficients \( C_i \) of various order \( N \) =5,10,15,20 by using eqn. (5a), (5b), (5c) and (5d)

\[ \text{Fig.1 Distribution of Gaussian points of order N=5} \]

\[ \text{Fig.2 Distribution of Gaussian points of order N=10} \]

\[ \text{Fig.3 Distribution of Gaussian points of order N=15} \]

\[ \text{Fig.4 Distribution of Gaussian points of order N=20} \]

4. Numerical Results

In this section, we consider the examples to show the present formulation may be applied to integrate any arbitrary function which cannot be evaluated even analytically. The some integrals are evaluated with the Generalized Gaussian quadrature rules up to order 20 which are shown in Table 1.

<table>
<thead>
<tr>
<th>Table 1 Evaluation of integrals with Generalized Gaussian quadratures</th>
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<tbody>
<tr>
<td>Exact value</td>
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<tr>
<td>---------------</td>
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<tr>
<td>(1) [ \int_{0}^{1} \int_{0}^{1} \int_{0}^{1} \frac{1}{\sqrt{1+z^2}} , dz , dy , dx = 0.784872216 ]</td>
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<tr>
<td>(2) [ \int_{0}^{1} \int_{0}^{1} \frac{\sqrt{x+y+z}}{\sqrt{1+z^2}} , dz , dy , dx = 1.205656863 ]</td>
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<tr>
<td>(3) [ \int_{0}^{1} \int_{0}^{1} \frac{\log(x+y+z)}{\sqrt{1+z^2}} , dz , dy , dx = -1.50163618 ]</td>
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\[
\iiint \frac{e^{y^2}}{\sqrt{1 + x^2 + y^2 + z^2}} \, dz \, dy \, dx = 0.6472258363
\]
\[
\iiint \frac{\sin(e)}{\sqrt{1 + x^2 + y^2 + z^2}} \, dz \, dy \, dx = 0.1078596350
\]

5. Conclusions

In this paper we derived Generalized Gaussian quadrature method for calculating integral over a cubic region \((x, y, z)/0 \leq x \leq 1, 0 \leq y \leq 1 and 0 \leq z \leq 1\). New Gaussian points and its weights are calculated of order \(N = 5, 10, 15, 20\). We have then evaluate the typical integrals governed by the proposed method. The results obtained are in excellent agreement with the exact value.

References


