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Research Article

Layerwise Theories for Cross-Ply Laminated Composite Beam

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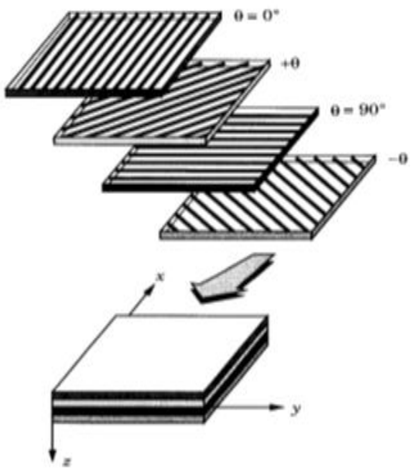
Abstract

Layerwise elementary theory of beam (ETB) and First order shear deformation theory (FSDT) for laminated composite beam are discussed together with their merits and demerits. The advanced beam theory is used for an accurate stress analysis of two-layered (90/0) cross-ply laminated composite beam having one of their edges rigidly clamped and other is free. Theory includes transverse shear deformation effect with shear correction factor. The elementary theory does not account the effect of transverse shear whereas First order shear deformation theory includes this effect. Principle of virtual work is used to obtain governing differential equation and boundary conditions in theory. Transverse shear stress can be obtained from constitutive relations and equilibrium equations as well, satisfying the stress free conditions at top and bottom surface of beam. The results are obtained for two layered (90/0) cross-ply laminated cantilever beam subjected to sinusoidal loading and compared with available results in literature.

Keywords: *Shear deformation; cross-ply laminated beam; First order shear deformation theory*

Introduction

A laminate is a collection of lamina stacked to achieve desired stiffness and thickness. The lamination scheme and material properties of individual lamina provide an added flexibility to designers to tailor the stiffness and strength of laminate to match the structural stiffness and strength requirements.



The theoretical concepts and analysis methods presented herein can help structural engineers in aerospace, civil and mechanical engineering industries to select suitable material for the best performance in a particular application.

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In classical plate theory, which is well known as elementary theory of beam (ETB), it is assumed that line which is normal to the neutral surface before deformation remain straight and normal to the neutral surface after deformation. This assumption results in under-estimation of deflection and over-estimation of natural frequencies and buckling loads. The theory is suitable for slender beams but not for thick or deep beams. since the theory neglects transverse shear deformation, it leads to less accurate results in the case of isotropic thick beams and more so in the case of laminated composite thick beams, where shear effects are significant. Timoshenko (Timoshenko, 1921) was the first to include the effects of rotatory inertia and shear deformation in the beam theory. In the early days the classical plate theory was extended for the analysis of the composite structures.

K.P. Saldatos and P. Watson (K.P. Saldatos and P. Watson, 1997) has given the methods of improving the stress analysis performance of one and two dimensional theories for laminated composites. Y.M.Ghugal and S.B.Shinde (Y.M.Ghugal and S.B.Shinde, 2011) provided a review of discrete layer shear deformation theory for flexure of thick cross-ply laminated composite beams. Zenkour (Zenkour, 1999) has developed higher order shear deformation beam theory. Analytical solution of theory is obtained using the Navier-like approach for simply supported boundary conditions.

Ghugal and Shimpi (Ghugal and Shimpi, 2001) provided a review of shear deformation theories for isotropic laminated plates. Reddy and Robbins (Robbins, 1994) has presented a review on theories and computational models for laminated composites. Levy and Stein (Levy, 1877; Stein, 1986) developed refined plate

theories expressing the displacement field in terms of trigonometric functions to represent the thickness effect and approximated the shear stress distribution through the thickness. Liu and Li (Liu D. and Li, 1996) presented an overall comparison of laminate theories based on displacement hypothesis emphasizing the importance of layer wise theories and also presented a series of quasi-layer wise theories. The equivalent single layer (ESL) theories are incapable to predict the piecewise distribution of inplane displacements. The transverse stresses predicted are erroneous and double-valued when obtained using constitutive relations. To remove these discrepancies in ESL theories, the use of layer wise theories for the analysis of thick laminates became necessary. Such layerwise theories are developed and used by Reddy (Reddy J. N, 1987), Lu and Liu (Lu X. and Liu D, 1992). A simple and easy to use layerwise shear deformation theory for flexural analysis of cross-ply laminated composite beams is developed more recently by Shimpi and Ghugal (Shimpi R. P. and Ghugal Y. M, 1999).
Literature study explains the need of layer wise analysis of laminated composite structural components in order to get more precise results in terms of transverse shear deformation, which is more pronounced in thick beams. In this paper, the layerwise first order shear deformation theory is presented, wherein linear differential equation is obtained in terms of transverse displacement. The theory satisfies the constitutive relationship in respect of transverse shear stress and shear strain in each layer and also interface shear stress continuity is satisfied.

Theoretical Formulations

The theoretical formulation of a cross-ply laminated beam based on certain kinematical and physical assumptions is presented. The variationally correct forms of differential equations and boundary conditions, based on the assumed displacement field are obtained using the principal of virtual work.

The beam under consideration consists of two layers: Layer 1 and layer 2.

Layer1 (90o layers) occupies the region:

$$0 \le x \le L; \quad -b/2 \le y \le b/2; \quad 0 \le z \le h/2 \tag{1}$$

Layer 2 (0o layers) occupies the region:

$$0 \le x \le L; \quad -b/2 \le y \le b/2; \quad -h/2 \le z \le 0 \tag{2}$$

Where x, y, z are Cartesian coordinates, L is the length, b is the width and h is the total depth of beam. The beam is subjected to transverse load of intensity q(x) per unit length of the beam. The beam can have any meaningful boundary conditions.

Displacement field

The displacement Field of present theory is of the form as given below.

$$u^{(1)}(x,z) = -(z-\alpha h)\phi_x \tag{3}$$

$$u^{(2)}(x,z) = -(z-\alpha h)\phi_x \tag{4}$$

$$w(x) = w(x) \tag{5}$$

Here u(1) and u(2) are the axial displacement components in the x direction, superscript 1 and 2 refer to layer 1 layer 2, w(x) is the transverse displacement in the z direction, ϕ_x is rotation of cross section from neutral axis.

Strain

Normal and transverse shear strains for layer 1 and layer 2.

$$\varepsilon_x^{(1)} = \frac{du^{(1)}}{dx} = -(z-\alpha h)\frac{d\phi}{dx} \tag{6}$$

$$\varepsilon_x^{(2)} = \frac{du^{(2)}}{dx} = -(z-\alpha h)\frac{d\phi}{dx} \tag{7}$$

$$\gamma_{zx}^{(1)} = \frac{du^{(1)}}{dz} + \phi = -\phi + \phi = 0 \tag{8}$$

$$\gamma_{zx}^{(2)} = \frac{du^{(2)}}{dz} + \phi = -\phi + \phi = 0 \tag{9}$$

Stresses

One dimensional constitutive laws are used to obtain the normal bending and transverse shear stresses for layer 1 and layer 2.

$$\sigma_x^{(1)} = E^{(1)}\varepsilon_x^{(1)} = -E^{(1)}(z-\alpha h)\frac{d\phi}{dx} \tag{10}$$

$$\sigma_x^{(2)} = E^{(2)}\varepsilon_x^{(2)} = -E^{(2)}(z-\alpha h)\frac{d\phi}{dx} \tag{11}$$

$$\tau_{ZX}^{(1)} = KG^{(1)}\left[\frac{dw}{dx} - \phi\right] \tag{12}$$

$$\tau_{ZX}^{(2)} = KG^{(2)}\left[\frac{dw}{dx} - \phi\right] \tag{13}$$

Governing equation and Boundary conditions

Using the expressions for strains and stresses (6)-(13) and principal of virtual work, variationally consistent differential equations and boundary conditions for the beam under consideration are obtained. The principle of virtual work when applied to the beam leads to:

$$b\int_{x=0}^{x=L}\int_{z=-h/2}^{z=0}\left(\sigma_x^{(1)}\delta\,\varepsilon_x^{(1)}\right)dx\,dz + b\int_{x=0}^{x=L}\int_{z=0}^{z=h/2}\left(\sigma_x^{(2)}\delta\,\varepsilon_x^{(2)}\right)dx\,dz - \int_{x=0}^{x=L} q\,\delta w\,dx = 0 \tag{14}$$

Where the symbol 'δ' denotes the variational operator.

Employing Green's theorem in above Eq. successively and collecting the coefficients of the primary variables (i.e. w), we obtain the governing equations and the associated boundary conditions. The governing equation is as follows:

$$-D\frac{d^2\phi}{dx^2}+AD_4\left(\phi-\frac{dw}{dx}\right)=0 \quad (15)$$

$$-AD_4\frac{d^2w}{dx^2}+Ad_4\frac{d\phi}{dx}=q_0\sin\frac{\pi x}{L} \quad (16)$$

Boundary Condition

$$-\left(E^{(1)}A_1+E^{(2)}B_1\right)\left[\frac{d^3w}{dx^3}\delta w\right]_0^l=0 \quad (17)$$

Or w is prescribed.
Where D is the constant defined as below.

$$D=(A_1+B_1)=\bar{D}E^{(2)}bh^3 \quad (18)$$

$$\bar{D}=\left(\frac{E^{(1)}}{E^{(2)}}\right)\left(\frac{1}{24}+\frac{\alpha}{4}+\frac{\alpha^2}{2}\right)+\left(\frac{1}{24}-\frac{\alpha}{4}+\frac{\alpha^2}{2}\right) \quad (19)$$

Layer 1 integration constant is as follows:

$$A_1=bh^3E^{(1)}\left(\frac{1}{24}+\frac{\alpha}{4}+\frac{\alpha^2}{2}\right) \quad (20)$$

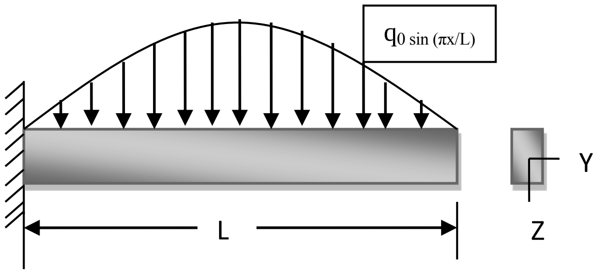
Layer 2 integration constant is as follows:

$$B_1=bh^3E^{(2)}\left(\frac{1}{24}-\frac{\alpha}{4}+\frac{\alpha^2}{2}\right) \quad (21)$$

Thus, the static behavior of beam is given by the solution of these variationally consistent governing differential equations and simultaneously satisfaction of the associated boundary conditions.

Illustrative Example

Example1: Cantilever beam with sinusoidal load-$q_{0\sin(\pi x/L)}$



Using governing differential equation 'φ' and 'w' is obtained

$$\phi=\frac{1}{D}\frac{qL^3}{\pi}\left[\frac{1}{\pi^2}\cos\frac{\pi x}{L}-\frac{x^2}{2L^2}+\frac{x}{L}-\frac{1}{\pi^2}\right]$$

$$w=\begin{bmatrix}\frac{q}{D}\frac{L^4}{\pi^4}\left[\sin\frac{\pi x}{L}-\frac{\pi^3}{6}\frac{x^3}{L^3}+\frac{\pi^3}{2}\frac{x^2}{L^2}-\frac{x\pi}{L}\right]\\+\frac{q}{AD_4}\frac{L^2}{\pi^2}\left[\sin\frac{\pi x}{L}-\frac{x\pi}{L}\right]\end{bmatrix}$$

Substituting expressions for w, the final expressions for axial displacements can be obtained

$$u_1=-\left(\frac{z}{h}-\alpha\right)\frac{h}{D}\frac{qL^3}{\pi}\left[\frac{1}{\pi^2}\cos\frac{\pi x}{L}-\frac{x^2}{2L^2}+\frac{x}{L}-\frac{1}{\pi^2}\right]$$

$$u_2=-\left(\frac{z}{h}-\alpha\right)\frac{h}{D}\frac{qL^3}{\pi}\left[\frac{1}{\pi^2}\cos\frac{\pi x}{L}-\frac{x^2}{2L^2}+\frac{x}{L}-\frac{1}{\pi^2}\right]$$

Substituting expressions for w, the final expressions for axial stresses can be obtained

$$\sigma_x^{(1)}=-E^{(1)}\left(\frac{Z}{h}-\alpha\right)\frac{h}{D}\frac{qL^2}{\pi}\left[-\frac{1}{\pi}\sin\frac{\pi x}{L}-\frac{x}{L}+1\right]$$

$$\sigma_x^{(2)}=E^{(2)}\left(\frac{Z}{h}-\alpha\right)\frac{h}{D}\frac{qL^2}{\pi}\left[-\frac{1}{\pi}\sin\frac{\pi x}{L}-\frac{x}{L}+1\right]$$

Expression for transverse shear stresses derived from the constitutive relationships τ_{zx}^{CR}

$$\tau_{ZX}^{(1)}=G^{(1)}\left[\frac{du^{(1)}}{dz}+\frac{dw}{dx}\right]=0$$

$$\tau_{ZX}^{(2)}=G^{(2)}\left[\frac{du^{(2)}}{dz}+\frac{dw}{dx}\right]=0$$

Expressions for transverse shear stress, τ_{zx}^{EQL} obtained from the equilibrium equations

$$\tau_{ZX}^{(1)}=E^{(1)}\left(\frac{z^2}{2}-\alpha hz-\frac{h^2}{8}-\frac{\alpha h^2}{2}\right)\frac{qL}{\pi D}\left[-\cos\frac{\pi x}{L}-1\right]$$

$$\tau_{ZX}^{(2)}=\begin{bmatrix}E^{(2)}\left(\frac{z^2}{2}-\alpha hz\right)\frac{qL}{\pi D}\left[-\cos\frac{\pi x}{L}-1\right]\\+E^{(1)}\left(-\frac{h^2}{8}-\frac{\alpha h^2}{2}\right)\frac{qL}{\pi D}\left[-\cos\frac{\pi x}{L}-1\right]\end{bmatrix}$$

Performance Analysis (Numerical Results):

The results for axial displacement, transverse displacement, axial and transverse stresses are presented in the following non dimensional form for the purpose of presenting the results in this work. Furthermore it may be noted that τ_{zx} and $\bar{\tau}_{zx}$ when obtained by constitutive relations indicated by τ_{zx}^{CR} and $\bar{\tau}_{zx}^{CR}$ when they are obtained by using equilibrium equations, are indicated by τ_{zx}^{EE} and $\bar{\tau}_{zx}^{EE}$

Table 1: Comparison of non-dimensional axial displacement ($\bar{u}$) (aspect ratio=S=4) For beams subjected to sinusoidal loading, q(x)

x/L	0.00		0.2		0.4		0.6		0.8		1.0	
Source / z/h	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**
0.5	0.00	0.00	-2.25	-2.25	-3.50	-3.50	-4.03	-4.03	-4.03	-4.03	-4.17	-4.17
0.4	0.00	0.00	-1.41	-1.41	-2.20	-2.20	-2.53	-2.53	-2.53	-2.53	-2.62	-2.62
0.3	0.00	0.00	-0.58	-0.58	-0.90	-0.90	-1.03	-1.03	-1.10	-1.10	-1.07	-1.07
0.2	0.00	0.00	0.25	0.25	0.39	0.39	0.45	0.45	0.47	0.47	0.47	0.47
0.1	0.00	0.00	1.09	1.09	1.70	1.70	1.95	1.95	2.02	2.02	2.02	2.02
0.0	0.00	0.00	1.93	1.93	3.00	3.00	3.45	3.45	3.56	3.56	3.57	3.57
0.0	0.00	0.00	1.93	1.93	3.00	3.00	3.45	3.45	3.56	3.56	3.57	3.57
-0.1	0.00	0.00	2.76	2.76	4.30	4.30	4.95	4.95	5.11	5.11	5.12	5.12
-0.2	0.00	0.00	3.60	3.60	5.60	5.60	6.44	6.44	6.65	6.65	6.67	6.67
-0.3	0.00	0.00	4.44	4.44	6.91	6.91	7.94	7.94	8.20	8.20	8.22	8.22
-0.4	0.00	0.00	5.27	5.27	8.21	8.21	9.44	9.44	9.75	9.75	9.77	9.77
-0.5	0.00	0.00	6.11	6.11	9.51	9.51	10.94	10.94	11.29	11.29	11.321	11.321

Table 2: Comparison of non-dimensional axial stress ($\bar{\sigma}_x$) (aspect ratio=S=4)

x/L	0.00		0.2		0.4		0.6		0.8		1.0	
Source / z/h	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**
0.5	-87.69	-87.69	-55.74	-55.74	-26.06	-26.06	-8.52	-8.52	-1.13	-1.13	0.00	0.00
0.4	-55.13	-55.13	-33.78	-33.78	-16.38	-16.38	-5.36	-5.36	-0.79	-0.79	0.00	0.00
0.3	-22.56	-22.56	-13.83	-13.83	-6.70	-6.70	-2.19	-2.19	-0.29	-0.29	0.00	0.00
0.2	9.99	9.99	6.12	6.12	2.97	2.97	0.97	0.97	0.12	0.12	0.00	0.00
0.1	42.56	42.56	26.08	26.08	12.56	12.56	4.13	4.13	0.54	0.54	0.00	0.00
0.0	0.75	0.75	46.04	46.04	22.33	22.33	7.30	7.30	0.96	0.96	0.00	0.00
0.0	0.30	0.30	1.84	1.84	0.89	0.89	0.29	0.29	0.03	0.03	0.00	0.00
-0.1	4.30	4.30	2.64	2.64	1.28	1.28	0.41	0.41	0.05	0.05	0.00	0.00
-0.2	5.61	5.61	3.43	3.43	1.66	1.66	0.54	0.54	0.07	0.07	0.00	0.00
-0.3	6.91	6.91	4.23	4.23	2.05	2.05	0.67	0.67	0.08	0.08	0.00	0.00
-0.4	8.21	8.21	5.03	5.03	2.44	2.44	0.79	0.79	0.10	0.10	0.00	0.00
-0.5	9.51	9.51	5.83	5.83	2.82	2.82	0.92	0.92	0.12	0.12	0.00	0.00

Table 3: Comparison of non-dimensional Transverse shear stress (τ_{zx}^{EE}) (aspect ratio=S=4)

x/L	0.00		0.2		0.4		0.6		0.8		1.0	
Source / z/h	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**	**ETB**	**FSDT**
0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
0.4	3.56	3.56	3.32	3.32	2.33	2.33	1.23	1.23	0.34	0.34	0.00	0.00
0.3	5.50	5.50	4.98	4.98	3.60	3.60	1.90	1.90	0.52	0.52	0.00	0.00
0.2	5.82	5.82	5.26	5.26	3.81	3.81	2.01	2.01	0.55	0.55	0.00	0.00
0.1	4.50	4.50	4.07	4.07	2.95	2.95	1.55	1.55	0.43	0.43	0.00	0.00
0.0	1.56	1.56	1.41	1.41	0.10	0.10	0.54	0.54	0.14	0.14	0.00	0.00
0.0	1.56	1.56	1.41	1.41	0.10	0.10	0.54	0.54	0.14	0.14	0.00	0.00
-0.1	1.38	1.38	1.25	1.25	0.90	0.90	0.47	0.47	0.13	0.13	0.00	0.00
-0.2	1.13	1.13	1.02	1.02	0.74	0.74	0.39	0.39	0.10	0.10	0.00	0.00
-0.3	0.82	0.82	0.74	0.74	0.53	0.53	0.28	0.28	0.07	0.07	0.00	0.00
-0.4	0.44	0.44	0.40	0.40	0.29	0.29	0.15	0.15	0.04	0.04	0.00	0.00
-0.5	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00

$$\bar{u} = \frac{E^{(1)}bu}{qh}, \bar{w} = \frac{100E^{(1)}bh^3 w}{qL^4}$$

$$\bar{\sigma}_x = \frac{b\sigma_x}{q}, \bar{\tau}_{zx} = \frac{b\tau_{zx}}{q}$$

A cantilever two- layered (90/ 0) composite beam, wherein layers 1 and 2 occupy the regions given by expressions (1) and (2), respectively, is considered for detailed numerical study. The beam is subjected to sinusoidal load $q_{(x)} = q_0 \sin\frac{\pi x}{L}$ acting in the z direction. The material of the beam layers is a carbon-fiber/epoxy unidirectional composite. The following has been assumed:

$$E^{(1)} = 1\times10^6 mpa, E^{(2)} = 25\times10^6 mpa,$$
$$G^{(1)} = 0.20\times10^6, G^{(1)} = 0.20\times10^6,$$
$$\bar{D} = 0.015641, \alpha = 0.23077.$$

where superscripts (1) and (2) refer to layers 1 and 2 respectively

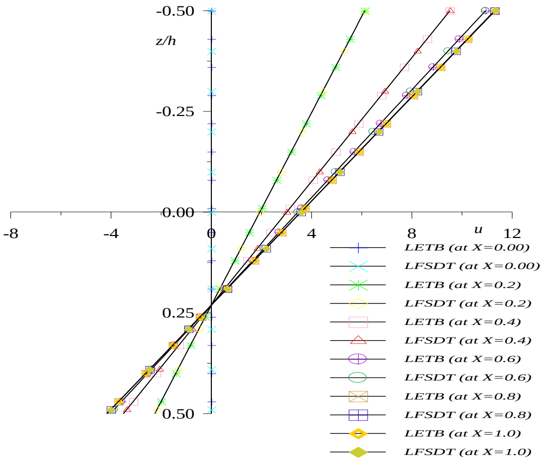


Fig. 1: Variation of axial displacement (u) through the thickness of cantilever beam when subjected to sinusoidal load for aspect ratio 4

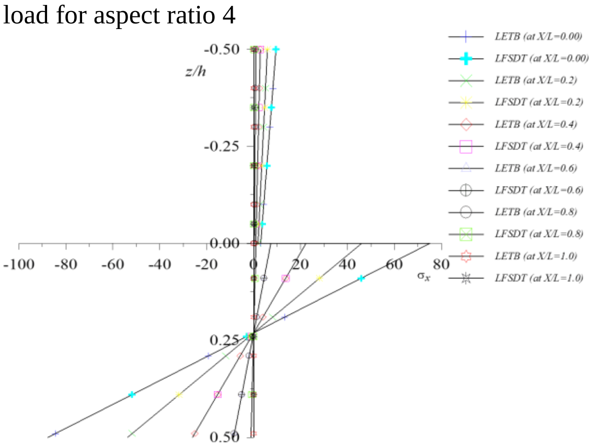


Fig 2: Variation of axial stress (σ_x) through the thickness of cantilever beam when subjected to sinusoidal load for aspect ratio 4

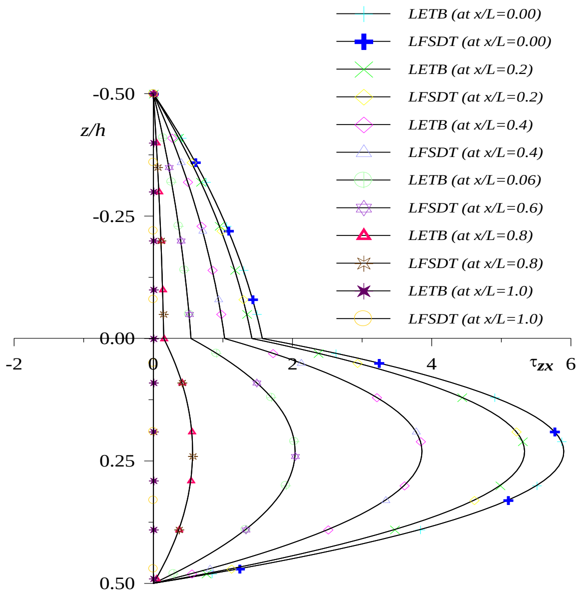


Fig 3: Variation of transverse shear stress (τ_{zx}^{EE}) through: the thickness of cantilever beam subjected to sinusoidal load and obtained using equilibrium equations for aspect ratio 4

Comparison of transverse displacement (w) (aspect ratio=S=4)

The results of maximum transverse displacement (w) for the various aspect ratio are presented here in Fig 4 for a cantilever beam subjected to sinusoidal load.
The results of cantilever beam in Example 1 subjected to sinusoidal load, for maximum non-dimensional axial displacement ($\bar{u}$), axial or normal bending stress ($\bar{\sigma}_x$) and transverse shear stress (τ_{zx}^{EE}) are presented in Table 1, Table 2 and Table 3.

Discussion of Results

The results of displacement and stresses in this paper are presented in following non- dimensional form for the purpose of comparison. The results obtained by FSDT for displacement and stresses are compared with the ETB. The results of maximum transverse displacement (w) for various aspect ratios for cantilever beam subjected to sinusoidal load are shown in fig 4. It is found that ETB underestimates the maximum transverse displacement as compared to FSDT in case of thick beams.

Conclusion

A discrete layer shear deformation theory is used for the static flexural analysis of cross ply laminated (90/0) beams. From the analysis, following conclusions are drawn.

1. The results of transverse displacement according to FSDT are in good agreement with those of ETB.
2. Transverse displacement values for FSDT converges to the ETB values with increase in aspect ratio. As

FSDT accounts the shear deformation effect, the values of transverse displacement increases with decrease in aspect ratio.

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Nomenclature

1. b = width of beam
2. D = Flexural rigidity
3. $\bar{D}$ = Modified flexural rigidity coefficient as defined in Appendix
4. $E^{(1)}, E^{(2)}$ = Young's module of layer 1 and layer 2, respectively
5. H = Depth (i.e. thickness) of beam
6. L = Span of the beam
7. S = Aspect ratio (i.e. ratio of span to depth of beam)
8. x, y, z = Rectangular coordinates
9. $\bar{u}$ = Non-dimensional axial displacement
10. w = Non-dimensional transverse displacement
11. $\bar{\sigma}_x$ = Non-dimensional axial stress
12. τ_{zx}^{EE} = Non-dimensional transverse shear stress obtained from the equilibrium equations

Abbreviations

Superscripts

CR- Constitutive relationships
EE- Equilibrium equations

Acronyms

ETB- Elementary theory of beam bending
FSDT- First-Order shear deformation theory