Influence of Soil Stiffness and of End Support Conditions of a Pile in its Buckling Force

Vlora Shatri, Luljeta Bozo, Burbuqe Shatri, and Bajram Shefkiu

Abstract

The aim of this work is to analyze the effects of soil stiffness and of pile end conditions into the buckling force of the pile fully embedded on ground and subject to only axial compression force. The pile buckling force is analyzed for the case when the modulus of soil reaction is \( k=0 \), \( k=\text{const} \) and \( k=n \). The software SAP 2000 is used for analysis.

Keywords: Pile, buckling force, buckling length, modulus of soil reaction.

1. Introduction

The phenomena of buckling is described as an instability of a completely straight column subject to an axial load when exceeding a certain value. Initially buckling was limited in defining/assessing the limiting value of the force beyond which a column becomes instable.

Resulting out of the nowadays researches it is found that buckling of piles usually occurs when the same are embeded in a very soft soils such as the soft clays, very incoherent sands, etc. Mathematically and experimentally is established that vulnerability to the buckling of piles considerably decreases out of lateral limitations causeed by the surrounding soil and due to this the buckling phenomena is in most of soil types neglected.

2. Analysis of pile buckling and determination of critical buckling force

The pile buckling problem is closely related to the problem of a beam on an elastic foundation; therefore with the aim of calculating the response of a vertical pile fully embedded on ground and subject to an external load, the pile will be treated as a beam of an elastic foundation (Matlock, H. and L. C. Reese)

The critical buckling force of a pile in an environment that offers resistance to the lateral displacement depends on:

- pile length, \( L \)
- soil stiffness, \( K \) and \( E_s \)
- Conditions of the end restraint of the pile with the pad and the soil.

Flexural pile stiffness, \( EI \).

Most of analitical methods used to analyze buckling of piles are based on the theory of soil reaction modulus. The therm soil reaction modulus represents pressure \( P \) per unit area of the contact surface between the loaded beam or slab and the soil it is supported by and on which the load is transmitted on. The coefficient of the soil reaction, \( k \), is the ratio between lateral soil pressure \( P \), on each determined node of the contact surface and the deflection, \( y \), caused by the respective force applied at that node:

\[
k = \frac{P}{y} \left[ \text{kN/m}^2 \right] \tag{1}
\]

2.1 Determination of pile buckling force for the case when \( k=0 \)

A pile of the following end conditions will be considered: pinned on the head and pinned on the tip (\( p-p \)). The equation of flexure of a pile subject to axial load is:

\[
E \cdot I \frac{d^4 y}{dx^4} + P \frac{d^2 y}{dx^2} = k \cdot y = 0 \tag{2}
\]

Substituting \( k=0 \) in equation (2) the following is obtained:

\[
E \cdot I \frac{d^4 y}{dx^4} + P \frac{d^2 y}{dx^2} = 0 \tag{3}
\]

Substituting \( a^2 = P/(EI) \), the differential equation (3) takes the following shape:

\[
\frac{d^4 y}{dx^4} + a^2 \frac{d^2 y}{dx^2} = 0 \tag{4}
\]

After a fourfold integration a differential equation of the following type will be obtained:

\[
y(x) = C_1 \cdot \sin \alpha \cdot x + C_2 \cdot \cos \alpha \cdot x \tag{5}
\]

*Corresponding author: Vlora Shatri

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The constants of integration are determined out of the end conditions of a pile.

\[ y(x=0) = 0 \quad C_2 = 0 \]  
\[ y(x=l) = 0 \quad C_1 \cdot \sin\alpha \cdot l = 0 \]  

(6)  
(7)

The equation (7) will be satisfied if \( C_1 = 0 \), and that is a trivial solution. A pile will not deflect and that means that \( y(x) = 0 \) for entire range of \( x \). This shows the position of the pile prior to buckling. Besides this the condition \( C_1 \cdot \sin\alpha \cdot l = 0 \) will be fulfilled for:

\[ \sin\alpha \cdot l = 0 \]  

(8)

This is the case when:

\[ \alpha = n \cdot \frac{\pi}{l} , \quad n = 1, 2, 3, \ldots \]  

(9)

Using the substitution \( \alpha^2 = P/(EI) \) we may write:

\[ P = n^2 \cdot \frac{\pi^2}{l^2} \cdot E \cdot I \]  

(10)

The smaller value of \( P \) for \( n=1 \) represents the critical buckling force of the \((p-p)\) \( P_{krit} \).

\[ P_{krit} = \frac{\pi^2}{l^2} \cdot E \cdot I \]  

(10)

In Figure 1, are given four basic cases of piles with equal length and different end conditions as well as the curves that represent the flexural shape of the piles in case when \( k=0 \), while the expressions for calculation of buckling forces are given in Table 1. For the all four cases of buckling, the buckling length is taken:

\[ l_0 = \beta \cdot l \]  

(11)

While the buckling force of the pile expressed through the buckling length is given:

\[ P_{krit} = \frac{\pi^2}{l_0^2} \cdot E \cdot I \]  

(12)

### Table 1

<table>
<thead>
<tr>
<th>Case</th>
<th>Formula</th>
<th>Notes</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pile pinned on the head – pinned on the tip</td>
<td>( P_{krit} = \frac{\pi^2}{l^2} \cdot E \cdot I )</td>
<td></td>
</tr>
<tr>
<td>Pile pinned on the head – fixed on the tip</td>
<td>( P_{krit} = \frac{\pi^2}{(0.7\cdot l)^2} \cdot E \cdot I )</td>
<td></td>
</tr>
<tr>
<td>Pile free on the head – fixed on the tip</td>
<td>( P_{krit} = \frac{\pi^2}{(2.0\cdot l)^2} \cdot E \cdot I )</td>
<td></td>
</tr>
<tr>
<td>Pile fixed on the head – fixed on the tip</td>
<td>( P_{krit} = \frac{\pi^2}{(0.5\cdot l)^2} \cdot E \cdot I )</td>
<td></td>
</tr>
</tbody>
</table>

#### 2.2 Determination of the pile buckling force when \( k=\text{const} \)

We will write again the buckling differential equation for the pile subject to axial compression force only:

\[ E \cdot I \cdot \frac{d^2y}{dx^2} + P \cdot \frac{d^2y}{dx^2} + k \cdot y = 0 \]  

(2)

End conditions will be:

\[ y(x=0) = 0, \quad y'(x=0) = 0 \]
\[ y(x=l) = 0, \quad y'(x=l) = 0 \]  

(13)

The solution of differential equation (13) will be:

\[ y(x) = C \cdot \sin \left( \frac{n \cdot \pi \cdot x}{l} \right) \]  

(14)

Solution of differential equation (2) for the force \( P \) will be:

\[ P_{krit} = n^2 \cdot \frac{\pi^2}{l^2} \cdot E \cdot I + \frac{1}{n^2} \cdot k \cdot l^2 \]  

(15)

where:

\[ P_{krit} = \frac{\pi^2}{l^2} \cdot E \cdot I \]  

(16)

The first part of the equation (15) corresponds to Euler’s equation for buckling of a column while the second part reflects contribution of lateral restraint caused by the soil. In equation (15) it is taken that the relation force – deflection is linear and that the modulus of horizontal reaction of soil is constant within the depth, a fact that for most of soils does not prevail.

For a pile embedded in an elastic medium, a general expression for critical buckling force is:

\[ P_{krit} = P_0 \left( n^2 + \frac{B}{n^2} \right) \]  

(17)

where \( n \) is the number of semi-waves of a sinusoidal curve in the deflected shape of the column and

\[ B = \frac{k \cdot l^2}{\pi^2 \cdot E \cdot I} \]  

(18)
It is easy to show that minimal value of \( P_{\text{crit}} \) corresponds to the value

\[
n = \left[ k \cdot (E \cdot I) \right]^{1/4} l / \pi \cdot E \cdot I = B^{1/4}
\]  

(19)

For the pile with end conditions such as free on the head-fixed on the tip, the Euler’s force is:

\[
P_e = \frac{\pi^4}{4 \cdot l^2} \cdot E \cdot I
\]  

(20)

We substitute \( n \) and \( P_e \) in equation (17), simplify and will obtain:

\[
P_{\text{ave}} = \frac{1}{2} \left( k \cdot E \cdot I \right)^{1/2}
\]  

(21)

The critical buckling length of the pile is obtained from the condition:

\[
\frac{\partial P}{\partial l} = 0
\]  

(22)

\[
l_0 = \pi \cdot \left[ \frac{E \cdot I}{k \cdot \left[ \pi \right]} \right]
\]  

(23)

From where it is seen that the buckling length or as it’s called, the critical length contains the two elements that are: the pile element expressed through the stiffness of a pile \( EI \), and the element soil, expressed through the design soil coefficient, \( k \). If the pile is to stiff comparing to the soil, then the buckling length takes relatively high values, that means that pile load will cause considerable deflection/displacement. In contrary, if the soil is o stiff comparing to the pile, then the buckling length of the pile would be relatively small. This way, it is possible that for the system pile-soil to (determine) the buckling length and from the solution of the given problem define the point in which the deflection of the pile is equal to zero, hence beyond (downwards) this point the deflection is so small so as it can be neglected. If the pile is smaller than the buckling length we call it a stiff pile, at the other side, a pile with the buckling length greater than the buckling length is called a slender pile. With other words, as greater the ratio between the pile stiffness and the soil stiffness the greater pile length is required (necessary) in order the pile to be considered flexible.

For calculation of critical buckling force of the pile pinned at both ends, it is possible that in some cases we use the in-equation (Davission, M. T.)

\[
P_{\text{ave}} \geq 2 \cdot \sqrt{k \cdot E \cdot I}
\]  

(24)

Particularly for greater values of \( l \), critical buckling force iscalculated sufficiently accurate according to the following expression (Shnell, W., Czerwenka, G.):

\[
P_{\text{ave}} \approx 2 \cdot \sqrt{k \cdot E \cdot I}
\]  

(25)

If we compare the value of critical buckling force of the pile, pinned at both ends, for \( k=\text{const} \), with a column of the same supporting conditions but that is not subject to lateral load along its length \( l \), \( (k=0) \), usually the critical buckling force is much greater than the buckling force of the column.

3. Pile buckling analysis by Davission’s Method

3.1 Determination of a pile buckling force for the case when \( k=\text{const} \).

If the differential equation (2) is divided by \( EI \) we will obtain:

\[
\frac{d^4 y}{dx^4} + \frac{P}{E \cdot I} \cdot \frac{d^2 y}{dx^2} + \frac{k}{E \cdot I} \cdot y = 0
\]  

(26)

Further solution of the differential equation (26) will be made in a way that all parameters within the equation are taken as being constant. The solution of the equation (26) will be obtained in a non-dimensional form if taken:

\[
R = \frac{E \cdot I}{k}, \quad Z = \frac{x}{R}, \quad \frac{Z_{\text{max}}}{R} = \frac{L}{R}
\]  

(27)

this way:

\( L \) support length of a pile
\( R \) relative stiffness factor
\( EI \) pile stiffness
\( Z \) non-dimensional coefficient of depth

Differential equation (26) will take this shape:

\[
\frac{d^4 y}{dz^4} + P \cdot R^2 \cdot \frac{d^2 y}{dz^2} + y = 0
\]  

(28)

Taken that

\[
U_{\text{ave}} = \frac{P_{\text{ave}} \cdot R^2}{E \cdot I}
\]  

(29)

\( U_{\text{ave}} \), is the axial force coefficient. We substitute the axial force coefficient in equation (29) and we obtain:

\[
\frac{d^4 y}{dz^4} + U \cdot \frac{d^2 y}{dz^2} + y = 0
\]  

(30)

The critical values of the axial force, \( U_{\text{crit}} \), are taken from the solution of the equation (30) according to \( U \), bay taking into account the end conditions and the pile length \( Z_{\text{max}} \) (Fig. 2). The \( L/R \) parameter is often used to distinguish between the stiff pile and flexible:

\[
L/R > 4: \quad \text{stiff}
\]

\[
2 < L/R \leq 4: \quad \text{mediate}
\]

\[
L/R < 4: \quad \text{flexible}
\]
3.2 Determination of buckling force of a pile for the case when $k = n_0 \cdot x$.

In the case when the soil reaction modulus is taken as $k = n_0 \cdot x$, buckling of the pile will be influenced by end conditions and especially the end condition on the head of the pile since the pile tends to deflect where the lowest soil reaction modulus is. We substitute in equation (2) $k = n_0 \cdot x$, then the equation will take this shape:

$$E \cdot I \frac{d^4 y}{dx^4} + P \frac{d^2 y}{dx^2} + n_0 \cdot x \cdot y = 0$$  \hspace{1cm} (31)

Substituting with:

$$T = \frac{E \cdot I}{n_0} , \quad Z = \frac{x}{T} , \quad Z_{\text{max}} = \frac{L}{T}$$  \hspace{1cm} (32)

Where:

$T$ the relative stiffness factor
$Z$ the depth non-dimensional coefficient
$Z_{\text{max}}$ maximum value of coefficient of depth

Equation (31) will take the following shape:

$$\frac{d^4 y}{dx^4} + P \frac{T^2}{E \cdot I} \frac{d^2 y}{dx^2} + Z \cdot y = 0$$  \hspace{1cm} (33)

If we denote with $V$, the coefficient of axial force $P \cdot T^2 / EI$, later on:

$$V_{\text{crit}} = \frac{P_{\text{crit}} \cdot T^2}{E \cdot I}$$  \hspace{1cm} (34)

By substituting the expression (34) in equation (33) we will have:

$$\frac{d^4 y}{dx^4} + V_{\text{crit}} \frac{d^2 y}{dx^2} + Z \cdot y = 0$$  \hspace{1cm} (35)

Equation (35) is solved for $V_{\text{crit}}$ as a function of $Z_{\text{max}}$, in a computational way, according to Davisson (1963). In Fig.3 is given the $V_{\text{crit}}$, in a function of $Z_{\text{max}}$, for a pile with these differing support conditions.

4. Calculation of buckling load with software SAP 2000

Linear analysis of buckling of piles with the use of SAP 2000 software requires instable shapes of structure due to the P-delta effect subject to a group of respective loads. The buckling analysis includes general solution of the problem of eigen values:

$$[K - \lambda G(r)] \Psi = 0$$  \hspace{1cm} (36)

where it is:

$K$ stiffness matrix,
$G(r)$ required geometrical stiffness (P-delta) due to the load vector $r$,
$\lambda$ diagonal matrix of eigen values, and
$\Psi$ the matrix that corresponds to the eigen vector (modal shape)

Each couple of eigen value – eigen vector represents a shape of buckling of the structure. There are shapes for buckling identified in series from 1 to $n$ that can be calculated by use of the software.

The eigen value $\lambda$ is called the buckling factor. The load $r$, is to be multiplied with this factor so as to cause buckling in a given shape. This factor could be taken as a safety factor too: if the buckling factor is greater than one, the given load should be increasing to cause buckling, if this factor is smaller than one, the load should be decreasing in order to prevent buckling. The buckling a factor could also be negative. This shows that the buckling will happen if the loads are opposite.
We can create various cases of linear buckling force. Combination of one or more modal loads and/or loads due to acceleration contained in a vector of loads (load vector) \( r \), should be specified in any case. Using this software, the number of buckling shapes could be also specified in order to find a tolerance of the convergence. It is always recommended to seek more than one buckling shape since the first ten shapes may have very similar buckling factors. A minimum of six buckling shapes are recommended out of what it is understood that they depend on the load.

With the aim of determining the pile buckling load according to software \textit{SAP 2000}, the pile is divided into \( n \) equal pieces of length of \( \Delta x \). The modulus of soil reaction is calculated for each node of the pile as in the Fig. 5.

### 5. Results and Discussion

#### 5.1 Influence of the soil stiffness in the buckling force in case when the soil stiffness increases linearly with the depth, \( k=k_0+n_h x \) and with \( k_0=\text{const.} \) according to \textit{SAP 2000}

In Fig. 6 are given the values of buckling forces calculated with the software \textit{SAP 200}, where for the purpose of analysis is adopted a reinforced concrete pile of the following end conditions: pinned on the head – pinned on the tip, and of stiffness of \( EI= 12120.9 \text{ kNm}^2 \), \( C=25/30 \), \( D=30 \text{ cm} \).

Diagrams of buckling force are given in fig.6 as a function of pile length \( L \). In the first case the stiffness is adopted \( k=100 \text{ kN/m}^2 \) while in the other two cases the stiffness is taken \( k=100 \text{ kN/m}^2 \) and \( n_h=100 \text{ kN/m}^3 \) as well as \( k=100 \text{ kN/m}^2 \) and \( n_h=500 \text{ kN/m}^3 \). From Fig. 6 we notice that buckling shapes differ between with an increase of a pile length and we also have an increase of buckling force as the value \( n_h \) increases.

#### 5.2 Effects of increased values of \( k_0 \) in the buckling force when \( n_h=\text{constant} \)

In Fig.7 are given the diagrams of buckling forces \( P_{\text{crit}} \) as a function of pile length \( L \), stiffness of \( 1212.9 \text{ kNm}^2 \), \( n_h=100 \text{ kN/m}^3 \), and \( k_0= 0, 100, 500 \) and 1000 \( \text{ t/m}^2 \). For these cases too, the buckling shapes vary with an increase of the pile length and the buckling forces increase with an increase of \( k_0 \) while \( n_h \) is kept constant. Comparing the given diagrams in Fig. 7 with those in Fig. 6 could be concluded that the values of buckling forces are bigger for cases when \( k=100 \text{ kN/m}^2 \) and \( n_h=0, 100 \) and 500 \( \text{ kN/m}^3 \) comparing with the cases for \( n_h=100 \text{ kN/m}^3 \) and \( k_0 = 0, 100, 500, \) and 1000\( \text{ t/m}^2 \).

#### 5.3 Influence of boundary conditions on the pile buckling force

In Fig. 8 are given the diagrams of critical buckling forces of a pile as a function of the pile length \( L \) for the case when \( k=100/\text{m}^2 \) and \( n_h=100/\text{m}^3 \) for two end conditions:

![Figure 4](image1.png)

**Figure 4** Division of pile on finite elements length of \( \Delta x \)

![Figure 5](image2.png)

**Figure 5** Discretization of the effect of interaction soil-pile. (a) Idealization of the structure of piles, (b) soil reaction modulus varying linearly and with the depth, (c) soil modulus varying proportionally with the depth (Chandrasekaran, 1974).
(p-p) – pinned on the head and pinned in the tip of the pile, and (F-F) – fixed in the head and fixed in the tip of the pile.

Values of critical buckling forces based on SAP 2000 software for the case when pile end support conditions pinned on the tip – pinned on the head when \( n_h=100\text{kN m}^3 \)

From Fig.8 it is noticed that the buckling force relatively decreases with an increase of a pile length. For a pile with the end conditions (F-F), the values of buckling forces are considerably bigger comparing to those of the pile with the end conditions (p-p).

![Figure 7](image1.png)

**Figure 7** Values of critical buckling forces based on SAP 2000 software for the case when pile end support conditions pinned on the tip – pinned on the head when \( n_h=100\text{kN m}^3 \)

![Figure 8](image2.png)

**Figure 8** Values of critical buckling force for a pile with end different conditions, when \( k=100\text{t/m}^2 \) and \( n_h=100\text{kN m}^3 \) based on SAP 2000 software

**Conclusions**

The modal shapes of buckling of piles analysed with SAP 2000 software are influenced by pile length since they differ between with an increase of the length of the pile.

The buckling force of a pile is influenced by the pile stiffness. The buckling forces are increased when \( n_h \)

values are increased while \( k \) remains constant. Buckling force values are also increased in cases when the \( n_h \) is kept constant and \( k \) increased. For a pile of the same length, the bigger buckling force is obtained when \( n_h \) remains constant and the \( k \) is increased as in the opposite case. Hence the soil stiffness directly influences the buckling force of the pile.

The end support conditions are also influencing the pile buckling force. The buckling force is maximal for the pile with the end conditions fixed-fixed (F-F) and minimal for the end conditions pinned-pinned (p-p).

If the pile is to stiff comparing with the soil, then the buckling length has relatively larger value, this means that the pile load will cause considerable deflections. In contrary if the soil is to stiff comparing to the pile, then the buckling length of the pile would be relatively shorter. Beside the above mentioned factors for the pile buckling force, the pile stiffness itself has its influence too.

**References**

Hetenyi, M. L. (1960) Beam on Elastic Foundation, University of Michigan Press,


Shnell, W., Czerwenka. G.: Einführung in die Rechennethoden des Leichtbaus. Bibliographisches Institut Mannheim, Band 2)