A Common Fixed Point Theorem on L-Fuzzy Metric Space by Biased Mappings of Type $R_m$

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Abstract

As already aware of the concepts of compatible mappings, compatible mappings of type α, compatible mappings of type β and weakly compatible, we are proving a common fixed point theorem in L-fuzzy metric space on four self-mappings using the concept of Biased Mappings of Type and the property C with a known fact that if a pair of maps is compatible or compatible of type α or compatible of type β then it is biased mapping of both type. This result could be further used to prove result on three self-mappings.

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1. Introduction

As it is well known to every mathematician working on fixed point theory that the notion of fuzzy sets was introduced by L.A. Zadeh (1965). Various concepts of fuzzy metric space were introduced Z.K.Deng(1982), M.A.Erceg(1979) O. Kaleva et al.(1984), I.Kramosil et al. (1975) and fixed point theorems were proved as in S.S.Chang et al.(1997), Y.J.Cho et al.(1998), J.Goguen(1967), V.Gregori et al.(2002). Using the idea of ℒ-fuzzy sets J.Goguen (1967) , R. Saadati et al. (2008, 2008, 2006) introduced the notion of ℒ-fuzzy metric space with the help of continuous t-norm and proved fixed point theorems for weakly commuting maps and weak compatible mappings. H.Adibi et al. (2006), Rajesh Shrivastava et al. (2011) also proved their results on ℒ-fuzzy metric space for compatible mappings of type (P) and weak compatible mappings respectively. Here we are proving the fixed point theorem on L-Fuzzy Metric Space by Biased Mappings of Type ($R_M$) the concept introduced by M.R.Singh at al.(2009) on four self-mappings.

2. Preliminaries

Def.2.1. Complete Lattice is a partially ordered set (L, ≤) in which all subsets have both supremum (join) and infimum (meet).

Def.2.2. J. Goguen (1967) Let L= ($L_{\leq} L$) be a complete lattice and U a non-empty set called universe. An L-fuzzy set A on U is defined as a mapping A: U→ L. For each u in U, A (u) represents the degree (in L) to which u satisfies A.

Lemma 2.3. G. Deschrijver et al.(2003) Consider the set L' = {[(x_1, x_2); (x_1, x_2) ∈ [0,1]^2 and x_1 + x_2 ≤ 1]} (x_1, x_2) ≤_{L'} (y_1, y_2) if and only if x_1 ≤ y_1 and x_2 ≥ y_2 for every (x_1, x_2), (y_1, y_2) ∈ L'.Then(L', ≤_{L'}) is complete lattice.

Def.2.4. A.T. Atanassov (1986) An intuitionistic fuzzy set A on a universe U is an object $A_{\in, \notin}$ on universe U. Let $\alpha_A(u)$, $\beta_A(u)$ be the membership degree and non-membership degree respectively of u in A and furthermore satisfy $\alpha_A(u) + \beta_A(u) ≤ 1$. A triangular norm Ton ([0, 1], ≤) is defined as an increasing, commutative, associative mappings $T:[0,1]^2 →[0,1]$ satisfying $T(1, x) = x \forall x ∈ [0,1]$. We define first $0_\in = \inf L$, $1_\in = \sup L$.

Def.2.5. A triangular norm (t-norm) on L is a mapping $J: L^2 → L$ satisfying the following conditions:

1. $(\forall x ∈ L) (J(x, 1_L) = x)$ (boundary condition);
2. $(\forall (x, y) ∈ L^2) (J(x, y) = J(y, x))$ (commutativity);
3. $(\forall (x, y, z) ∈ L^3) (J(x, J(y, z) = J(J(x, y), z))$ (associativity);
4. $(\forall (x, y, z) ∈ L^3) (x ≤_L x’ and y ≤_L y’ → J(x, y) ≤_L J(x’, y’)$ (monotonicity);

A t-norm can also be defined recursively as an (n+1)-ary operation $(\forall n ∈ N \setminus \{0\})$ by $J^n = J$ and

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Proof: For every \( \lambda \in L \backslash \{0, 1\} \), there exists \( \lambda \in L \backslash \{0, 1\} \) such that 
\[
E_{\lambda,M}(x_n, x_m) \leq E_{\lambda,M}(x_{n+1}, x_{m+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m).
\]

Hence the sequence \( \{x_n\} \) is Cauchy.

Lemma 2.13. Let \( (X, M, T) \) be an \( L\)-fuzzy metric space. If \( \lim_{n \to \infty} M(x_n, x_{n+1}, k^n t) \) for some \( k>1 \) and \( n \in \mathbb{N} \), then \( \{x_n\} \) is a Cauchy sequence.

From lemma 2.12, for every \( \mu \in L \backslash \{0, 1\} \), there exists \( \lambda \in L \backslash \{0, 1\} \) such that 
\[
E_{\lambda,M}(x_n, x_m) \leq E_{\lambda,M}(x_{n+1}, x_{m+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m).
\]

Hence the sequence \( \{x_n\} \) is a Cauchy sequence.

Lemma 2.14. Let A and S be two mappings from \( L\)-fuzzy metric space \( (X, M, T) \) into itself and \( \{x_n\} \) be a sequence in \( X \) such that \( \lim_{n \to \infty} A x_n = \lim_{n \to \infty} S x_n \) for some \( u \in X \). Then the mapping A and S are said to be

i. Weakly commuting if \( M(Ax, S x, t) \geq M(Ax, S x, t) \) for all \( x \) in \( X \).

ii. Compatible if \( \lim_{n \to \infty} M(A x_n, S x_n, t) = M(A x, S x, t) \).

iii. Compatible of type (A) if \( M(A x_n, S x_n, t) \geq M(x_n, t) \) for all \( x \) in \( X \).

iv. Compatible of type (P) if \( \lim_{n \to \infty} M(A x_n, S x_n, t) = M(x_n, t) \) for all \( x \) in \( X \).
Def. 2.16. We say that $L$-fuzzy metric space $(X, M, T)$ has a property $C$, if it satisfies the following condition $M(x, y, t) = C$ for all $t > 0$ implies $C = 1_L$.

I. Main Theorem

Theorem 3.1. Let $A$, $B$, $S$ and $T$ be self-maps of a $L$-fuzzy metric space $(X, M, T)$ with a property $C$ into itself and $x$ and $y$ be $L$-fuzzy

1. $A(Y)UB(X) \subseteq S(X) \cap T(X)$;
2. $M(Ax, By, t) \geq M(Sx, Ty, t/\alpha)$;
3. One of each pair $[A, S]$ and $[B, T]$ is continuous;
4. The pair $[A, S]$ is $S$-biased of type $(R_M)$ and $[B, T]$ is $T$-biased of type $(R_M)$.

If one of the range space of $A$, $B$, $S$ and $T$ is complete subspace of $X$ then $A$, $B$, $S$ and $T$ have common fixed point $x$.

Proof: Since from (i) $A(Y) \subseteq T(X)$ thus we choose any arbitrary point $x_0 \in X$ which exists a point $x_1 \in X$ so that $Ax_0 = T_{x_1}$. Similarly as $B(X) \subseteq S(X)$, thus for any arbitrary point $x_1 \in X$ there exists a point $x_2 \in X$ so that $Bx_1 = S_{x_2}$. Therefore by induction we construct a sequence $\{x_n\}$ such that $y_{2n+1} = Tx_{2n+1} = Ax_{2n}$ and $y_{2n+2} = Sx_{2n+1} = Bx_{2n+2}$ for $n \in \{0, 1, 2, \ldots \}$.

We first prove $\{y_n\}$ is a Cauchy sequence in $(X, M, T)$. $M(y_{2n+1}, y_{2n+2}, t) \geq M(Sx_{2n}, Tx_{2n+1}, t/\alpha) = M(y_{2n-1}, y_{2n}, t/\alpha)$

$M(y_{2n}, y_{2n-1}, t/\alpha) \geq M(y_{2n-1}, y_{2n-2}, t/\alpha)$

Therefore for every $\mu \in L \setminus \{0, 1_L\}$ there exists $\delta \in L \setminus \{0, 1_L\}$ such that $E_{\mu, \delta}(y_m, y_{m+1}) \leq E_{\mu, \delta}(y_{m-1}, y_m) + E_{\mu, \delta}(y_{m+1}, y_{m+2}) + \ldots + E_{\mu, \delta}(y_{m-1}, y_{m+1})$

$\leq \alpha E_{\mu, \delta}(y_m, y_{m+1}) + \alpha E_{\mu, \delta}(y_{m-1}, y_m) + \ldots + \alpha E_{\mu, \delta}(y_{m+1}, y_{m+2}) = \alpha^m E_{\mu, \delta}(y_0, y_1) \sum_{i=0}^{m-1} \alpha^i \to 0$.

Thus by lemma (2.11), $\{y_n\}$ is a Cauchy sequence. Now suppose that $S(X)$ is complete subspace of $X$ then $\{y_n\}$ converges to a point $u \in S(X)$. Since $S(X)$ is also a subsequence of $\{y_n\}$ which converges to $u \in S(X)$. Therefore the sequence $\{y_n\}$ converges to a point $u \in S(X)$. Thus $Ax_{2n}, Bx_{2n+2}, Sx_{2n+1}$ and $Tx_{2n+1}$ converges to $u \in S(X)$. As $S(X)$ is complete subspace of $X$, thus there exist a point $v \in X$ such that $Sv = u$.

Put $y = x_{2n+1}$ and $x = v \in (i)$, we get $M(Av, Bx_{2n+1}, t) \geq M(Sv, Tx_{2n+1}, t/\alpha)$.

Take the limit as $n \to \infty$, we get $M(Av, u, t) \geq M(Sv, u, t/\alpha) = 1_L$.

This implies $A \equiv S$. Now suppose that $S$ is continuous then $SSx_{2n} \to Su$.

Also $[A, S]$ is $S$-biased of type $(R_M)$.

$\Rightarrow \lim_{n \to \infty} M(SSx_{2n}, Sx_n, t) \geq_{\delta} \lim_{n \to \infty} M(ASx_{2n}, Ax_n, t) \forall t > 0$.

Now to prove $Su = u$, for this put $x = Sx_{2n}, y = x_{2n+1}$ in (ii).

$M(ASx_{2n}, Bx_{2n+1}, t) \geq M(SSx_{2n}, Tx_{2n+1}, t/\alpha)$.

Now take the limit as $n \to \infty$

$\Rightarrow \lim_{n \to \infty} M(SSx_{2n}, Bx_{2n+1}, t) \geq_{\delta} \lim_{n \to \infty} M(SSx_{2n}, Tx_{2n+1}, t/\alpha)$.

$\Rightarrow M(Su, u, t) \geq M(Su, u, t/\alpha)$.

Therefore we have $M(Su, u, t/\alpha) \geq M(Su, u, t/\alpha)^2 \geq 1_L$...

$\Rightarrow M(Su, u, t) = M(Su, u, t/\alpha^2)$.

But by lemma (2.9) we know that $M(x, y, t)$ is non-decreasing with respect to $t$ for all $x, y \in X$.

$\Rightarrow M(Su, u, t/\alpha) \leq M(Su, u, t/\alpha^2)$.

Therefore $M(Su, u, t/\alpha) = C$ for all $t > 0$. Since $M(Su, u, t/\alpha)$ has the property $(C)$, it follows that $C = 1_L$ that is $Su = u$.

Now to prove $Au = Su$.

Put $x = u$ and $y = x_{2n+1}$ in (ii), we get $M(Au, Su, u) = M(Au, Bx_{2n+1}, t) \geq M(BTx_{2n+1}, t/\alpha)$.

$\Rightarrow M(Au, Su, u) \geq M(Bu, u, t/\alpha) = 1_L$.

Thus $Au = Su$.

Therefore $Au = u$. $\Rightarrow Au = u$.

Now to prove $Bu = Tu$.

Suppose that $T$ is continuous, thus $TTx_{2n+1} \to Tu$. As $[B, T]$ is $T$-biased type of $(R_M)$

$\Rightarrow \lim_{n \to \infty} M(BTx_{2n+1}, Bx_{2n+1}, t) \geq_{\delta} \lim_{n \to \infty} M(BTx_{2n+1}, Bx_{2n+1}, t)$

To prove $Tu = u$, for this take $x = x_{2n}, y = x_{2n+1}$ in (ii).

$M(Ax_{2n}, BTx_{2n+1}, t) \geq M(SSx_{2n}, TTx_{2n+1}, t/\alpha)$.

$M(BTx_{2n+1}, AX_{2n+1}, t) \geq M(TTX_{2n+1}, Sx_{2n}, t/\alpha)$.

Now take the limit as $n \to \infty$

$M(BTx_{2n+1}, Bx_{2n+1}, t) \geq_{\delta} \lim_{n \to \infty} M(BTx_{2n+1}, Bx_{2n+1}, t)$

$\Rightarrow \lim_{n \to \infty} M(BTx_{2n+1}, Sx_{2n}, t/\alpha) \geq \lim_{n \to \infty} M(BTx_{2n+1}, Bx_{2n+1}, t)$

Then as proved above and using the lemma (2.9) that $M(x, y, t)$ is non-decreasing with respect to $t$ for all $x, y \in X$, we have

$\Rightarrow M(Tu, u, t/\alpha) \leq M(Tu, u, t/\alpha^2)$.

Therefore $M(Tu, u, t/\alpha) = C$ for all $t > 0$. Since $M(Tu, u, t/\alpha)$ has the property $(C)$, it follows that $C = 1_L$ that is $Tu = u$.

Thus $Tu = u$.

To prove $Bu = Tu$, put $x = x_{2n}, y = u$ in (ii), we get $M(Ax_{2n}, Bu, t) \geq M(SSx_{2n}, Tu, t/\alpha)$.

Take limit as $n$ tends to $\infty$ we get $M(Bu, u, t) \geq M(Su, u, t/\alpha) = 1_L$.

Thus $Bu = Tu$.

Therefore $Tu = u$.
This implies A, B, S and T have a common fixed point u ∈ X.

**Corollary 3.2.** Let A, S and T be self-maps of a ℋ-fuzzy metric space (X, ℋ, J) with a property C into itself and for α ∈ (0, 1) satisfying
i. A(X) ⊆ S(X) ∩ T(X);
ii. ℋ(Ax, Ay, t) ≥ ℋ(Sx, Ty, t/α);
iii. One of each pair [A, S] and [A, T] is continuous;
iv. The pair [A, S] is 𝕀-biased of type (R_M) and [A, T] is ℍ-biased of type (R_M).

If one of the range space of A, S and T is complete subspace of X then A, S and T have common fixed point in X.

**Proof:** To prove the result take B = A in the theorem 3.1.

**Corollary 3.3.** Let A, B and T be self-maps of a ℋ-fuzzy metric space (X, ℋ, J) with a property C into itself and for α ∈ (0, 1) satisfying
i. A(X) ⊆ B(X) ∩ T(X);
ii. ℋ(Ax, By, t) ≥ ℋ(Bx, Ty, t/α);
iii. One of each pair [A, T] and [B, T] is continuous;
iv. The pair [A, T] is ℍ-biased of type (R_M) and [B, T] is 𝕀-biased of type (R_M).

If one of the range space of A, B and T is complete subspace of X then A, B and T have common fixed point in X.

**Proof:** To prove the result take S = T in the theorem 3.1.

**References**


