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Research Article

Hamming Index of Class of Graphs

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Abstract

Let A(G) be the adjacency matrix of a graph G. The rows of A(G) corresponding to a vertex v of G, denoted by s(v) is the string which belongs to Z_2^n*, a set of n-tuples. The Hamming distance between the vertices u and v is the number of positions in which s(u) and s(v) differ. The Hamming index of a graph G is the sum of the Hamming distances between all pairs of vertices of G. In this paper we obtain the Hamming index of certain class of graphs.*

Keywords: *Hamming distance, Hamming index. Adjacency matrix, Mathematics Subject Classification: 05C99*

1. Introduction

[1]Let Z_2 ={0, 1} and (Z_2 +) be the additive group where + denotes addition modulo 2. For any positive integer n, $Z_2{}^n = Z_2 \text{ X } Z_2 \text{ X } \dots \text{ X } Z_2$ (n factors)
$= \{(x_1, x_2, \dots x_n \mid x_1, x_2, \dots x_n \in Z_2)\}$
Thus every element of $Z_2{}^n$ is an n-tuple $(x_1, x_2, \dots x_n)$ written as $x = x_1, x_2, \dots x_n$ where every x_i is either 0 or 1 and is called a string or word. The number of 1's in $x =_1, x_2, \dots x_n$ is called the *weight* of *x* and is denoned by wt(*x*).

Let $x = x_1, x_2, \dots x_n$ and $y = y_1, y_2, \dots y_n$ be the elements of $Z_2{}^n$. Then the sum *x* + *y* is computed by adding the corresponding components of *x* and *y* under addition modulo2. That is $x_i + y_j = 0$ if $x_i = y_j$ and $x_i + y_j = 1$ if $x_i \neq y_j$

The *Hamming distance* $H_d(x, y)$ between the strings $x = x_1 x_2, \dots x_n$ and $y = y_1 y_2, \dots y_n$ is the number of i's such that $x_i \neq y_i$, $1 \leq i \leq n$.

Thus $H_d(x, y)$ = Number of positions in which *x* and *y* differ = wt(*x*+*y*)
Example: Let *x* = 01001 & *y* = 11010 and *x* + *y* = 10011,

Therefore $H_d(x, y)$ = wt(*x*+*y*) = 3

A graph *G* is called a *Hamming graph* [3,6] if each vertex $v \in V(G)$ can be labeled by a string of a fixed length s(v) such that $H_d(s(v), s(u)) = d_G(u, v)$ for all $u, v \in V(G)$ where $d_G(u, v)$ is the length of shortest path joining *u* and *v* in *G*.

Hamming graphs are known as an interesting graph family in connection with the error-correcting codes and association schemes. For more details one can refer [S. Bang et al,2008, V. Chepoi,1988, W. Imrich et al,1997, S. Klavzar et al, 2005, H. S. Ramane et al, 2013, R. Squer et al, 2001].

Motivated by the work on Hamming graphs in this we introduce and study the Hamming Index of a graph in association with its adjacency.

2. Preliminaries

Let G be an undirected graph without loops and multiple edges. Let $V(G) = \{v_1, v_2, \dots v_n\}$, $E(G) = \{e_1, e_2, \dots e_n\}$ be the vertex set and edge set of G respectively. The vertices adjacent to the vertex *v* are called the *neighbours* of *v*. The *degree* of a vertex v, denoted by *degv* is the number of neighbours of *v*.

The *adjacency matrix* of G is a square matrix $A(G) = [a_{ij}]$ of order *n*

where $a_{ij} = 1$ if the vertex v_i is adjacent to v_j

= 0 otherwise.

Rows of *A*(*G*) represents the strings which belongs to $Z_2{}^n$. The Row of *A* (*G*) corresponding to the vertex $v \in V(G)$is the string denoted by *s*(*v*).

Definition 2.1: *Hamming distance* between the vertices *u* and *v* of *G* denoted by $H_d(u, v)$ is defined as $H_d(u, v) = H_d(s(u), s(v))$

Definition 2.2: The *Hamming Index*[10] of a graph *G*, denoted by *H*(*G*) is defined as,

$H(G) = \sum H_d(v_i, v_j)$ $1 \leq i < j \leq n$

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Hamming index is a graph invariant.

Ex. :

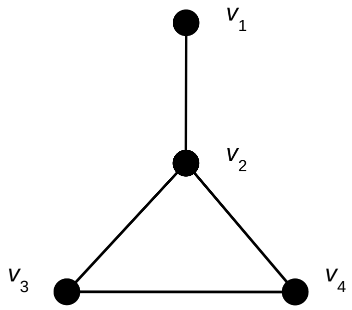

Fig. 1: Graph G

For a graph G of Fig. 1, the adjacency matrix is

$$A(G)=\begin{bmatrix}0&1&0&0\\1&0&1&1\\0&1&0&1\\0&1&1&0\end{bmatrix}$$

and the strings are $s(v_1)=0100$, $s(v_2)=1011$ $s(v_3)=0101$ $s(v_4)=0110$

$H_d(v_1,v_2)=4$ $H_d(v_1,v_3)=1$ $H_d(v_1,v_4)=1$

$H_d(v_2,v_3)=3$ $H_d(v_2,v_4)=3$ $H_d(v_3,v_4)=2$

Therefore $H(G)$ = 4+1+1+3+3+2 = 14

3. Hamming Index of certain class of graphs

Definition: Consider two graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2$. Then the graph whose vertex set $n_1 \mathrm{U} n_2$ and the edge set $m_1 \mathrm{U} m_2$ is called the union of G_1 and G_2 and is denoted by $G_1 \mathrm{U}\, G_2$.

Theorem 3.1: Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then Hamming index of union of G_1 and G_2 is,

$$H(G_1 \mathrm{U} \mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+2(n_1m_2+n_2m_1)$$

Proof:

Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then then the adjacency matrix of union of G_1 and G_2 is,

$$A\left(G_1 \mathrm{U} \mathrm{G}_2\right)=\begin{bmatrix}A(G_1) & O\\ O & A(\mathrm{G}_2)\end{bmatrix}$$

where $A(G_1)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 respectively, O is the null matrix.

Therefore,

$$H(G_1 \mathrm{U} \mathrm{G}_2)$$

$$=\sum_{1\le i<j\le n_1} H_{\mathrm{d}}(u_i,v_j)+\sum_{n_1\le i<j\le n_1+n_2} H_{\mathrm{d}}(u_i,v_j)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} H_{\mathrm{d}}(u_i,v_j)$$

$$= H(G_1)+H(G_2)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} \deg(u_i)+\deg(u_j)$$

$$=$$

$$H(G_1 \mathrm{U} \mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+2(n_1m_2+n_2m_1)$$

Definition: Consider two graphs $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$. The join of G_1 and G_2, denoted by $G_1 + G_2$, is $G_1 \mathrm{U} G_2$ along with each vertex of G_1 joined to every vertex of G_2 by an edge.

Theorem 3.2: Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then Hamming index of $G_1 + G_2$ is,

$$H(G_1+\mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+n_1n_2(n_1+n_2)-2n_1m_2-2n_2m_1$$

Proof:

Let $G_1(n_1, m_1)$ and $G_2(n_2, m_2)$ be two graphs then the adjacency matrix of join of G_1 and G_2 is,

$$A\left(G_1+\mathrm{G}_2\right)=\begin{bmatrix}\mathrm{A}(G_1) & J\\ J & \mathrm{A}(G_2)\end{bmatrix}$$

where $A(G_2)$ and $A(G_2)$ are adjacency matrices of G_1 and G_2 respectively and

$$J=\begin{bmatrix}1 & 1\ldots1\\ 1 & 1\ldots1\\ . & .\quad .\quad .\\ 1 & 1\ldots1\end{bmatrix}$$

Therefore,

$$H(G_1+\mathrm{G}_2)=\sum_{1\le i<j\le n_1+n_2} H_d(u_i,v_j)$$

$$=\sum_{1\le i<j\le n_1} H_{\mathrm{d}}(u_i,v_j)+\sum_{n_1\le i<j\le n_1+n_2} H_{\mathrm{d}}(u_i,v_j)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2} H_{\mathrm{d}}(u_i,v_j)$$

$$= H(G_1)+H(G_2)+\sum_{i=1}^{n_1}\sum_{j=n_1+1}^{n_1+n_2}\{n_1+n_2-[\deg(u_i)+\deg(u_j)]\}$$

$$\therefore \quad H(G_1+\mathrm{G}_2)=H(\mathrm{G}_1)+H(\mathrm{G}_2)+n_1n_2(n_1+n_2)-2n_1m_2-2n_2m_1$$

Definition: Let G be a graph with vertices $v_1, v_2, \ldots v_n$. A graph G^+ is obtained from G by inserting new vertices $u_1, u_2, \ldots u_n$, and joining u_i to v_i by an edge, i=1, 2, . . . n

Theorem 3.3: Let C_n *be cycle* on n vertices then Hamming index of C_n^+ is,

$$H(C_n^+) = \left[\frac{4n-5}{n-2}\right] H(C_n)$$

Proof:

Let C_n *be cycle* on n vertices then the adjacency matrix of C_n^+ is

$$A\left(C_n^+\right) = \begin{bmatrix} A(\mathrm{C_n}) & I \\ I & O \end{bmatrix}$$

where $A(C_n)$ is the adjacency matrix of C_n, I is the Identity matrix of order n and O is the null matrix

Therefore,

$$H(C_n^+) = \sum_{1 \le i < j \le 2n} H_d(u_i, v_j)$$

$$= \sum_{1 \le i < j \le n} H_\mathrm{d}(u_i, v_j) + \sum_{n+1 \le i < j \le 2n} H_\mathrm{d}(u_i, v_j) + \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j)$$

$$= \sum_{u,v \in C_n} 2 + H_\mathrm{d}(u,v) + \sum_{n+1 \le i < j \le 2n} H_\mathrm{d}(u_i, v_j) + \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j)$$

- - - - - (1)

Where,

i)

$$\sum_{n+1 \le i < j \le 2n} H_d(u_i, v_j) = 2({}^nC_2)$$

ii)

$$\sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) = \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) \quad \textit{for pair of } (u_i, v_j) \textit{ adjacent pairs}$$

$$+ \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) \quad \textit{for pair of } (u_i, v_j) \textit{ non-adjacent pairs}$$

$$\sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) = \text{hamming distances between n - adjacent pairs} = 4(\mathrm{n})$$

$$\sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) = \text{hamming distance between } (\mathrm{n}^2 - \mathrm{n}) \text{ non - adjacent pairs}$$

= Hamming distance between $2n$ pairs with common neighbour

+Hamming distance between $[(n^2\text{-n})\text{-}2n]$-pairs with non common neighbor= $2(2n) + 4(n^2\text{-}3\text{n})$

Substituting in eqn(1),

$$H(C_n^+) =$$

$$\left[H(C_n) + 2\,{}^nC_2\right] + 2\,{}^nC_2 + \left[4n + 2(2n) + 4(n^2 - 3n)\right]$$

$$= H(C_n) + 6n^2 - 6n$$

$$= 2n(n-2) + 6n^2 - 6n$$

[H. S. Ramane et al, 2013]

= 2n(4n-5

$$= \left[\frac{4n-5}{n-2}\right] H(C_n)$$

Theorem 3.4: Let K_n *be complete graph* on n vertices then Hamming index of K_n^+ is,

$$H(K_n^+) = H(K_n) + n^3 + n^2$$

Proof:

Let K_n *be complete graph* on n vertices then the adjacency matrix of K_n^+ is

$$A\left(K_n^+\right) = \begin{bmatrix} A(\mathrm{K_n}) & I \\ I & O \end{bmatrix}$$

where $A(\mathrm{K_n})$ is the adjacency matrix of $\mathrm{K_n}$, I is the Identity matrix of order n and O is the null matrix

Therefore,

$$H(K_n^+) = \sum_{1 \le i < j \le 2n} H_d(u_i, v_j)$$

$$= \sum_{1 \le i < j \le n} H_\mathrm{d}(u_i, v_j) + \sum_{n+1 \le i < j \le 2n} H_\mathrm{d}(u_i, v_j) + \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j)$$

$$= \sum_{u,v \in C_n} 2 + H_\mathrm{d}(u,v) + \sum_{n+1 \le i < j \le 2n} H_\mathrm{d}(u_i, v_j) + \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j)$$

- - - - - (1)

Where,

i)

$$\sum_{n+1 \le i < j \le 2n} H_d(u_i, v_j) = 2({}^nC_2)$$

ii)

$$\sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) = \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) \quad \textit{for pair of } (u_i, v_j) \textit{ adjacent pairs}$$

$$+ \sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) \quad \textit{for pair of } (u_i, v_j) \textit{ non-adjacent pairs}$$

$$\sum_{i=1}^{n} \sum_{j=n+1}^{2n} H_\mathrm{d}(u_i, v_j) = \text{hamming distances between n - adjacent pairs} = (\mathrm{n}+1)(\mathrm{n})$$

$$\sum_{i=1}^{n}\sum_{j=n+1}^{2n} H_d(u_i, v_j) = \text{hamming distance between } (n^2 - n) \text{ non-adjacent pairs} = (n-1)(n^2-n)$$

Substituting in eqn(1),

$$H(K_n^+) =$$
$$\left[H(K_n) + 2\,{}^nC_2\right] + 2\,{}^nC_2 + \left[n(n+1) + (n-1)(n^2-n)\right]$$
$$= H(K_n) + n^3 + n^2$$

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